

MODULI OF WEIGHTED STABLE ELLIPTIC SURFACES AND INVARIANCE OF LOG PLURIGENERA

KENNETH ASCHER AND DORI BEJLERI

ABSTRACT. This is the third paper in a series of work on weighted stable elliptic surfaces — elliptic fibrations with section and marked fibers each weighted between zero and one. Motivated by Hassett’s weighted pointed stable curves, we use the log minimal model program to construct compact moduli spaces parameterizing these objects. Moreover, we show that the domain of weights admits a wall and chamber structure, describe the induced wall-crossing morphisms on the moduli spaces as the weight vector varies, and describe the surfaces that appear on the boundary of the moduli space. The main technical result is a proof of invariance of log plurigenera for slc elliptic surface pairs with arbitrary weights.

1. INTRODUCTION

Elliptic fibrations are ubiquitous in mathematics, and the study of their moduli has been approached from many directions; e.g. via Hodge theory [HL02] and geometric invariant theory ([Mir81], [Mir80]). At the same time, moduli spaces often have many geometrically meaningful compactifications leading to different birational models. This leads to rich interactions between moduli theory and birational geometry.

The compact moduli space $\overline{\mathcal{M}}_g$ of genus g stable curves and its pointed analogue $\overline{\mathcal{M}}_{g,n}$ is exemplary. Studying the birational geometry of the moduli space of stable curves by varying the moduli problem has been a subject of active research over the past decade known as the *Hassett-Keel program* (see [FS13] for a survey). Our hope is to produce one of the first instances of this line of study for surfaces (see also [LO16] which initiates a similar line of study for quartic K3 surfaces). This paper, along with our work in [AB16a] and [AB16b], continues a study of the birational geometry of the moduli space of *stable elliptic surfaces* initiated by La Nave [LN].

One particular instance of the birational geometry of $\overline{\mathcal{M}}_{g,n}$ is developed by Hassett in [Has03]. He constructs various compactifications $\overline{\mathcal{M}}_{g,\mathcal{A}}$ of the moduli space of *weighted pointed curves* (see Section 9.1). These compact moduli spaces parameterize degenerations of genus g curves with marked points weighted by a vector \mathcal{A} . A natural question is: what happens to the moduli spaces as one varies the weight vector? Among other things, Hassett proves that there are birational morphisms $\overline{\mathcal{M}}_{g,\mathcal{B}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ when $\mathcal{A} \leq \mathcal{B}$ (Theorem 9.1). Furthermore, there is a wall-and-chamber decomposition of the space of weight vectors \mathcal{A} — inside a chamber the moduli spaces are isomorphic and there are explicit birational morphisms when crossing a wall.

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Hassett's space is the one dimensional example of a moduli spaces of *stable pairs* – pairs (X, D) of a variety along with a divisor satisfying having mild singularities and satisfying a positivity condition (see Definition 3.6). In this case, the variety is a curve C with at worst nodal singularities, the divisor is a weighted sum $D = \sum a_i p_i$ of smooth points on the curve, and one requires $\omega_C(D)$ to be an ample line bundle.

In this paper, we use stable pairs in higher dimensions to construct analogous compactifications of the moduli space of elliptic surfaces where the pair is given by an elliptic surface with section as well as \mathcal{A} -weighted marked fibers. The outcome is a picture for weighted surfaces completely analogous to that of $\overline{\mathcal{M}}_{g,\mathcal{A}}$.

In general, the story of compactifications of moduli spaces in higher dimensions is quite intricate and relies on the full power of the minimal model program. Many fundamental constructions have been carried out over the past few decades (e.g. [KSB88], [Ale94], & [KP15]). Although stable pairs have been identified as the right analogue of stable curves in higher dimensions, it has proven difficult to find explicit examples of compactifications of moduli spaces in higher dimensions (see [Ale06] for some examples), and thus we take as one goal of this paper to establish a wealth of examples of compact moduli spaces of surfaces that illustrate both the difficulties, as well as methods necessary to overcome them.

More specifically, for admissible weights \mathcal{A} , we construct and study $\mathcal{E}_{v,\mathcal{A}}$: the compactification by stable pairs of the moduli space of $(f : X \rightarrow C, S + F_{\mathcal{A}})$, where $f : X \rightarrow C$ is an elliptic surface with chosen section S , marked fibers weighted by \mathcal{A} , and fixed volume v . By \mathcal{A} admissible, we mean the moduli problem is non-empty.

Theorem 1.1 (see Theorem 4.4). *For admissible weights \mathcal{A} , there exists a moduli pseudofunctor of \mathcal{A} -weighted stable elliptic surfaces (see Definition 4.1) of volume v so that the main component $\mathcal{E}_{v,\mathcal{A}}$ is representable by a finite type separated Deligne-Mumford stack.*

To construct $\mathcal{E}_{v,\mathcal{A}}$ as an algebraic stack, we use the notion of a family of stable pairs given by Kovács-Patakfalvi in [KP15] and the construction of the moduli stack of stable pairs therein. However, representability of our functor does not follow immediately as we include the additional the data of the map $f : X \rightarrow C$ (see Section 4.1). Furthermore, the correct deformation theory for stable pairs has not yet been settled. As we are interested in the global geometry of the moduli space in this paper, we circumvent this issue by working exclusively with the normalization of the moduli stack. By the results of Appendix A, this amounts to only considering the functor on the subcategory of normal varieties.

Theorem 1.2 (see Theorem 8.4 and Theorem 8.6). *The moduli space $\mathcal{E}_{v,\mathcal{A}}$ is proper. Its boundary parametrizes \mathcal{A} -broken elliptic surfaces (see Definition 5.9 and Figure 1).*

This is one of the main results of the paper. As with the previous theorem, it does not follow immediately from known results about stable pairs because of the data of the map $f : X \rightarrow C$. Rather, we prove Theorem 1.2 by explicitly describing in Section 8 an algorithm for stable reduction that produces, as a limit, a stable pair as well as a map to a nodal curve. This is a generalization of the work of La Nave in [LN]. The main input is the use of twisted stable maps of Abramovich-Vistoli to produce limits of fibered surface pairs as discussed

in [AB16b] as well as previous results in [AB16a] and [LN], that describe the steps of the minimal model program on a one parameter family of elliptic surfaces.

Following Hassett, it is natural to ask how the moduli spaces change as we vary \mathcal{A} . The strategy in [Has03] is to understand how the objects themselves change as one varies \mathcal{A} , and then prove a strong vanishing theorem which guarantees that the formation of the relative log canonical sheaf commutes with base change. This ensures that the process of producing an \mathcal{A} -stable pointed curve from a \mathcal{B} -stable pointed curve with $\mathcal{A} \leq \mathcal{B}$ is functorial in families and leads to reduction morphisms on moduli spaces and universal families.

In [AB16a], we carried out a complete classification of the relative log canonical models of elliptic surface fibrations, and thus have completed the study of how the fibers of the objects themselves change as we vary the weight vector (see Section 5 and Theorem 5.3).

We prove an analagous base change theorem which implies that the steps of the minimal model program described in Section 5 are functorial in families of elliptic surfaces. The main technical tool is a vanishing theorem (Theorem 7.1) which relies on a careful analysis of the geometry of broken slc elliptic surfaces. We do not expect this vanishing theorem to hold in full generality for other classes of slc surfaces.

Theorem 1.3 (Invariance of log plurigenera, see Theorem 7.1 and Theorem 7.8). *Let $\pi : (X \rightarrow C, S + F_{\mathcal{B}}) \rightarrow B$ be a family of \mathcal{B} -broken stable elliptic surfaces (Definition 5.9) over a reduced base B . Let $0 \leq \mathcal{A} \leq \mathcal{B}$ be such that the divisor $K_{X/B} + S + F_{\mathcal{A}}$ is π -nef and \mathbb{Q} -Cartier. Then $\pi_* \mathcal{O}_X(m(K_{X/B} + S + F_{\mathcal{A}}))$ is a vector bundle on B whose formation is compatible with base change $B' \rightarrow B$ for $m \geq 2$ divisible enough.*

The main difficulty in the above theorem, and in the theory of stable pairs in general, is that smooth varieties will degenerate into non-normal varieties with several irreducible components. In higher dimensions these *slc* varieties can become quite complicated, e.g. see Figure 1 for a \mathcal{B} -broken elliptic surfaces that can appear in the limit of such a degeneration. Note in particular the map $f : X \rightarrow C$ is not equidimensional; there are irreducible components of X contracted to a point by f .

These components were first observed in the work of La Nave [LN] and were coined *pseudoelliptic* surfaces. They are the result of contracting the section of an elliptic surface. In fact La Nave noticed in the study of stable reduction for elliptic surfaces with no marked fibers ($\mathcal{A} = 0$), that a component of the section of f is contracted by the minimal model program if and only if the corresponding component of the base nodal curve C needs to be contracted to obtain a stable curve. We generalize this (Proposition 5.11) to the case of marked fibers and as a result obtain a morphism to Hassett space by forgetting the elliptic surface and remembering only the base weighted curve:

Theorem 1.4 (See Corollary 9.3). *There are forgetful morphisms $\mathcal{E}_{\mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$.*

Next we identify a wall and chamber decomposition of the space of admissible weights \mathcal{A} which describes at which \mathcal{A} does a one parameter family of broken elliptic surfaces undergo birational transformations leading to different objects on the boundary of the moduli stack.

In Section 8 we classify three types of birational transformations leading to three types of walls:

- there are W_I walls coming from the relative log minimal model program for the map $f : X \rightarrow C$ at which singular fibers change;
- there are W_{II} walls where a component of the section contracts to form a pseudoelliptic surface;
- there are W_{III} walls where an entire component of a broken elliptic surface may contract onto a curve or point.

Type W_I and W_{III} transformations result in divisorial contractions of the total space of a family of elliptic surfaces while type W_{II} result in small contractions which must then be resolved by a log flip. La Nave constructed this log flip explicitly and we show that this construction leads to a log flip of the universal family (see Section 10 and Figure 13). Putting this all together, our main theorem may be summarized as follows:

Theorem 1.5. *Let $\mathcal{A}, \mathcal{B} \in \mathbb{Q}^r$ be weight vectors such that $0 \leq \mathcal{A} \leq \mathcal{B} \leq 1$. We have the following:*

- (1) *If \mathcal{A} and \mathcal{B} are in the same chamber, then the moduli spaces and universal families are isomorphic.*
- (2) *If $\mathcal{A} \leq \mathcal{B}$ then there are reduction morphisms $\mathcal{E}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$ on moduli spaces which are compatible with the reduction morphisms on the Hassett spaces:*

$$\begin{array}{ccc} \mathcal{E}_{v,\mathcal{B}} & \longrightarrow & \mathcal{E}_{v,\mathcal{A}} \\ \downarrow & & \downarrow \\ \overline{\mathcal{M}}_{g,\mathcal{B}} & \longrightarrow & \overline{\mathcal{M}}_{g,\mathcal{A}} \end{array}$$

- (3) *The universal families are related by a sequence of explicit divisorial contractions and flips $\mathcal{U}_{v,\mathcal{B}} \dashrightarrow \mathcal{U}_{v,\mathcal{A}}$ such that the following diagram commutes:*

$$\begin{array}{ccc} \mathcal{U}_{v,\mathcal{B}} & \dashrightarrow & \mathcal{U}_{v,\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathcal{E}_{v,\mathcal{B}} & \longrightarrow & \mathcal{E}_{v,\mathcal{A}} \end{array}$$

More precisely, across W_I and W_{III} walls there is a divisorial contraction of the universal family and across a W_{II} wall the universal family undergoes a log flip.

The precise descriptions of the various wall crossing morphisms described above are given in Theorem 9.4, Corollary 8.7, Proposition 10.7, Theorem 10.1 and Proposition 10.4.

Now we will describe the objects that appear the boundary of $\mathcal{E}_{v,\mathcal{A}}$. While the minimal model program lends itself to an algorithmic approach towards finding minimal birational representatives of an equivalence class, it generally does *not* lead to an explicit stable reduction process as prevalent in $\overline{\mathcal{M}}_{g,n}$. However, using the minimal model program in addition to the theory of twisted stable maps developed by Abramovich-Vistoli [AV97], we are able to run an

explicit stable reduction process, and classify precisely what objects live on the boundary of our moduli spaces. This is inspired by the work of [LN]. The idea is that an elliptic fibration $f : X \rightarrow C$ with section S can be viewed as a rational map from the base curve to $\overline{\mathcal{M}}_{1,1}$, the stack of stable pointed genus 1 curves. One can use this to produce a birational model of f which can then be studied using twisted stable maps. The outcome is a compact moduli spaces of *twisted* fibered surface pairs studied in [AB16b] which forms the starting point of our analysis of one parameter degenerations in $\mathcal{E}_{v,\mathcal{A}}$.

Combining these one parameter degenerations produced by twisted stable maps with the wall crossing transformations discussed above, we identify in Section 8 the boundary objects parametrized by $\mathcal{E}_{v,\mathcal{A}}$:

Theorem 1.6. *[see Theorem 8.6] The boundary of the proper moduli space $\mathcal{E}_{v,\mathcal{A}}$ parametrizes \mathcal{A} -broken stable elliptic surfaces (see Definition 5.9) which are pairs $(f : X \rightarrow C, S + F_{\mathcal{A}})$ coming from a stable pair $(X, S + F_{\mathcal{A}})$ with a map to a nodal curve C such that:*

- X is an slc union of elliptic surfaces with section S and marked fibers, as well as
- chains of pseudoelliptic surfaces of type I and II (Definitions 5.7 and 5.8) contracted by f with marked pseudofibers (Definition 5.5).

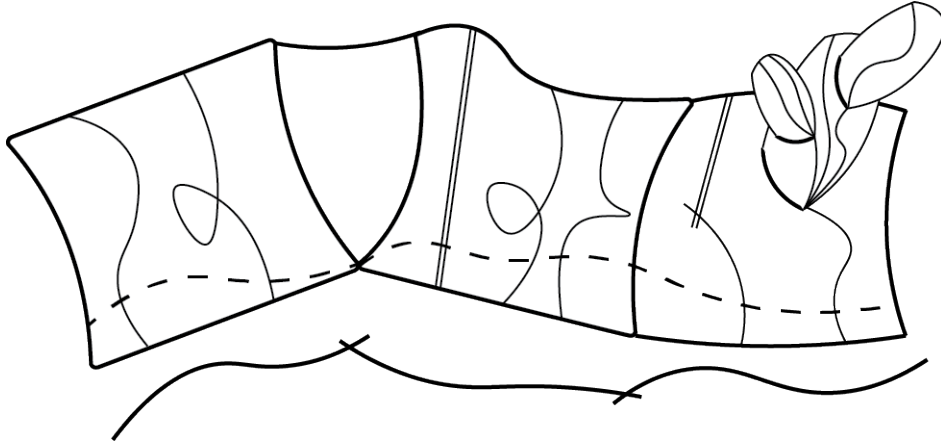


FIGURE 1. An \mathcal{A} -weighted broken elliptic surface.

Finally, in the appendix, we prove that in certain situations the normalization of an algebraic stack is uniquely determined by its values on normal base schemes (Proposition A.7) and that a morphism between normalizations of algebraic stacks can be constructed by specifying it on the category of normal schemes (Proposition A.6). This material is probably well known but we include it here for lack of a suitable reference.

1.1. An example. We illustrate the main results in a specific example. Figure 2 depicts the central fiber of a particular one parameter stable degeneration of a rational elliptic surface with twelve marked nodal fibers, ten of which are marked with coefficient 1 and the other two with coefficient α , as the coefficient α varies. The arrows depict the directions of the morphisms between the various models of the total space of the degeneration.

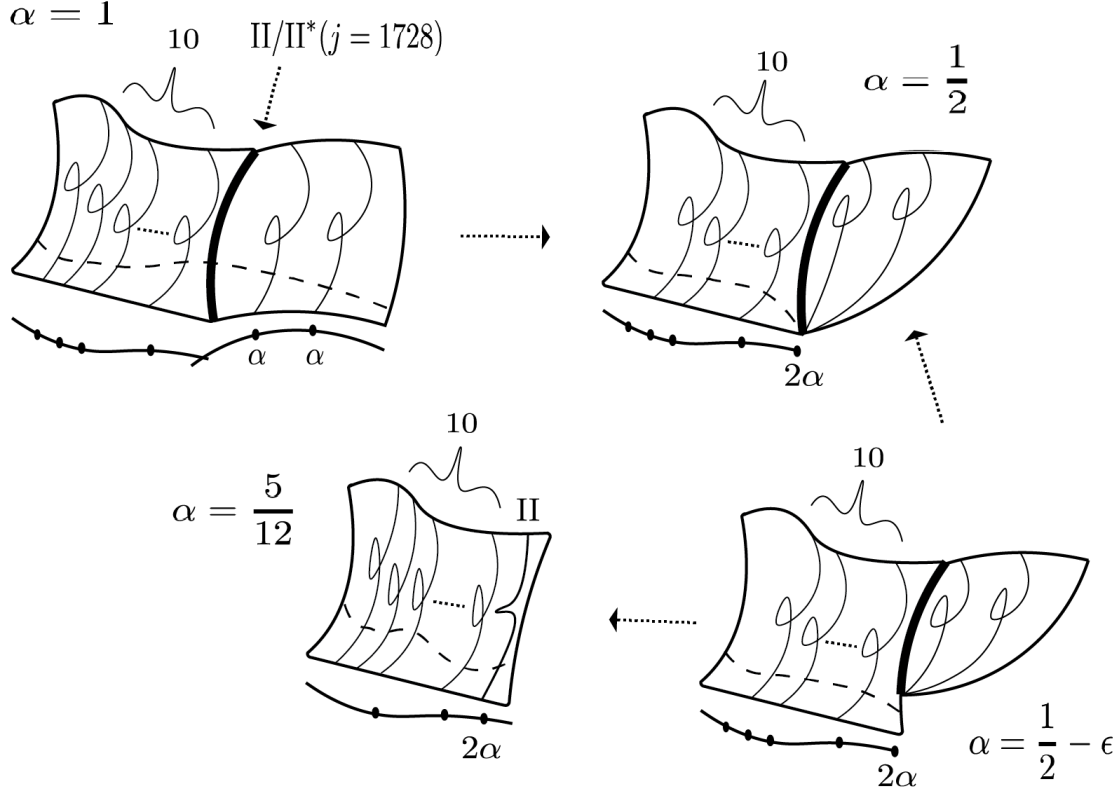


FIGURE 2. The wall crossing transformations on the central fiber of a stable degeneration of a rational elliptic surface.

The central fiber breaks up into a union of two components glued along twisted fibers of type II and II^* , one containing 10 marked nodal fibers with coefficient 1 and a type II twisted fiber and the other containing two nodal fibers marked with coefficient α and a type II^* twisted fiber. At $\alpha = 1/2$ the section of the second component contracts to form a pseudoelliptic surface. At $\alpha = 1/2 - \epsilon$ for any small enough $\epsilon > 0$, this contraction of the section is a log flipping contraction of the total space of the degeneration and a flip results in a different stable model. Finally at $\alpha = 5/12$ the whole pseudoelliptic component contracts to a point yielding an elliptic surface with 10 nodal fibers marked with coefficient 1 and a type II Weierstrass fiber with coefficient 2α . Each surface maps to the corresponding Hassett stable base curve as depicted.

1.2. Applications and further work. A simple corollary of the results in this paper is a classification of the singularities of stable degenerations of smooth elliptic surfaces. Combining Theorem 1.6 with the results of [AB16a] on singularities of log canonical models of elliptic surfaces (see also Section 5) as well as Proposition 5.17 we obtain the following:

Corollary 1.7. *Let $\mathcal{X}^0 \rightarrow \mathcal{C}^0 \rightarrow \Delta^0$ be a family of smooth relatively minimal elliptic surfaces over the punctured disc $\Delta^0 = \Delta \setminus \{0\}$ and with a fixed section and all singular fibers marked by a nonzero coefficient. Then after a base change of Δ^0 , the family can be*

extended to $\mathcal{X} \rightarrow \mathcal{C} \rightarrow \Delta$ such that the central fiber $X \rightarrow C$ is a broken elliptic surface. Each component of X is an elliptic or pseudoelliptic surface with only quotient singularities and the singularities are all rational double points except along type II, III and IV fibers. In particular, the normalization X^ν has klt singularities.

In upcoming work we will apply the results of this paper in the case when X is a rational or K3 surface. In particular, when X is rational, we relate our work to Miranda [Mir81], Heckman-Looijenga [HL02], as well as a few other spaces that have appeared in the literature. We show that the spaces $\mathcal{E}_{v,\mathcal{A}}$ and the wall crossing transformations described above interpolate between the various compactifications of moduli spaces of elliptic surfaces in the literature. Furthermore, as $\mathcal{E}_{v,\mathcal{A}}$ is modular, the explicit description of the boundary can be used to describe the boundaries of non-modular compactifications such as GIT models and compactifications of period domains.

Finally, we remark on our choice of boundary divisor. We fix the coefficient of the section to be 1. This is the key reason that the base curve of a stable elliptic surface is a Hassett stable curve (see Proposition 5.11). On the other hand, it is the reason for the formation of pseudoelliptic surfaces which leads to interesting yet complicated behavior across type W_{II} and W_{III} walls.

We also mark the *reduced* divisor underlying a scheme theoretic fiber (Definition 2.10). Theorem 5.3 then implies that the log canonical model interpolates between twisted fibers at coefficient 1 and Weierstrass fibers at coefficient 0, which allows us to use twisted stable maps as in [AB16b]. On the other hand, taking the scheme theoretic fiber as the marked divisor *always* leads to Weierstrass fibers.

Nevertheless, it would be interesting to extend our results to the case where the coefficient of S varies and the scheme theoretic fibers are marked. For example, when the coefficient of S is very small, one should obtain a compactification of a moduli space of Weierstrass fibrations by equidimensional slc elliptic fibrations. This particular situation in the case of K3 surfaces is studied by Brunyate in [Bru15].

1.3. Previous results. La Nave [LN] used twisted stable maps of Abramovich-Vistoli to prove properness of the moduli stack parameterizing elliptic surfaces in Weierstrass form via explicit stable reduction. He computes the stable models of one parameter families of elliptic surfaces in Weierstrass form. Roughly, given an elliptic surface $f : X \rightarrow C$ with section S , the Weierstrass form is obtained by contracting all components of the singular fibers of $f : X \rightarrow C$ that do not meet the section S . We will make repeated use of his work throughout. In our setting, this corresponds to the case of $\mathcal{E}_{\mathcal{A}}$ where $\mathcal{A} = 0$.

Brunyate [Bru15], described the KSBA stable pair limits of elliptic K3s with marked divisor $D = \delta S + \sum_{i=1}^{24} \epsilon F_i$, where $0 < \delta \ll \epsilon \ll 1$, the divisor S is a section, and the F_i are the 24 singular fibers.

In [Ale15], Alexeev provided another generalization of Hassett's picture for $\overline{\mathcal{M}}_{g,\mathcal{A}}$ to surfaces. He constructed reduction morphisms for the compact moduli spaces of *weighted hyperplane arrangements* – the moduli space parametrizing the union of hyperplanes in projective space.

Deopurkar in [Deo15] also suggested an alternate compactification of the moduli space of elliptic surfaces by admissible covers of the stacky curve $\overline{\mathcal{M}}_{1,1}$. It would be interesting to compare his space to those discussed here and in [AB16b].

We work over \mathbb{C} .

1.4. Outline.

- Sec. 2 (Pg. 9) We give background on elliptic surfaces.
- Sec 3 (Pg. 11) We give background on stable pairs, and recall some results about their moduli spaces.
- Sec 4 (Pg. 14) We give a preliminary definition for the objects that appear on the boundary of our moduli spaces, define a moduli functor, and prove that it is algebraic.
- Sec 5 (Pg. 17) We recall results on log canonical models of elliptic fibered surface pairs from [AB16a], introduce pseudoelliptic surfaces, prove that the base curve of a stable elliptic surface is a weighted stable curve, and classify log canonical contractions of elliptic surfaces.
- Sec 6 (Pg. 23) We give preliminaries on the log minimal model program and vanishing theorems.
- Sec 7 (Pg. 27) We prove a strong vanishing theorem for slc elliptic surfaces that implies invariance of log plurigenera.
- Sec 8 (Pg. 38) We prove a stable reduction theorem for $\mathcal{E}_{v,\mathcal{A}}$ to obtain properness, also enabling us to also give an explicit description of the surface on the boundary of our moduli space. In the process we describe a wall and chamber decomposition of the space of admissible weights.
- Sec 9 (Pg. 45) We construct reduction morphisms on our moduli space and universal family, and show that these morphisms are compatible with Hassett's reduction morphisms.
- Sec 10 (Pg 48) We show that along certain types of walls, the universal family undergoes a log flip.

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2. PRELIMINARIES ON ELLIPTIC SURFACES

We begin with general definitions, properties, and results on elliptic surfaces that will be important for describing and understanding the geometry of their moduli spaces.

2.1. Standard elliptic surfaces. We point the reader to [Mir89] for a detailed exposition on the theory of elliptic surfaces.

Definition 2.1. An irreducible **elliptic surface with section** $(f : X \rightarrow C, S)$ is an irreducible surface X together with a surjective proper flat morphism $f : X \rightarrow C$ to a proper smooth curve and a section S such that:

- (1) the generic fiber of f is a stable elliptic curve, and
- (2) the generic point of the section is contained in the smooth locus of f .

We say $(f : X \rightarrow C, S)$ is **standard** if all of S is contained in the smooth locus of f .

This definition differs from the usual definition of an elliptic surface in that we only require the generic fiber to be a *stable* elliptic curve.

Definition 2.2. A **Weierstrass fibration** $(f : X \rightarrow C, S)$ is an elliptic surface with section as above, such that the fibers are reduced and irreducible.

Definition 2.3. A surface is **semi-smooth** if it only has the following singularities:

- (1) 2-fold normal crossings (locally $x^2 = y^2$), or
- (2) pinch points (locally $x^2 = zy^2$).

Definition 2.4. A **semi-resolution** of a surface X is a proper map $g : Y \rightarrow X$ such that Y is semi-smooth and g is an isomorphism over the semi-smooth locus of X .

Definition 2.5. An elliptic surface is called **relatively minimal** if it is semi-smooth and there is no (-1) -curve in any fiber.

Note that a relatively minimal elliptic surface with section is standard.

If $(f : X \rightarrow C, S)$ is a standard elliptic surface then there are finitely many fiber components not intersecting the section. We can contract these to obtain an elliptic surface with all fibers reduced and irreducible:

Definition 2.6. If $(f : X \rightarrow C, S)$ is a standard elliptic surface then the Weierstrass fibration $f' : X' \rightarrow C$ with section S' obtained by contracting any fiber components not intersecting S is the **Weierstrass model** of $(f : X \rightarrow C, S)$. If $(f : X \rightarrow C, S)$ is relatively minimal, then we refer to $f' : X' \rightarrow C$ as the **minimal Weierstrass model**.

Definition 2.7. The **fundamental line bundle** of a standard elliptic surface $(f : X \rightarrow C, S)$ is $\mathbb{L} := (f_* N_{S/X})^{-1}$ where $N_{S/X}$ denotes the normal bundle of S in X . For $(f : X \rightarrow C, S)$ an arbitrary elliptic surface, we define $\mathbb{L} := (f'_* N_{S'/X'})^{-1}$ where $(f' : X' \rightarrow C, S')$ is a semi-resolution.

Since $N_{S/X}$ only depends on a neighborhood of S in X , the line bundle \mathbb{L} is invariant under taking a semi-resolution or the Weierstrass model of a standard elliptic surface. Therefore \mathbb{L} is well defined and equal to $(f'_* N_{S'/X'})^{-1}$ for $(f' : X' \rightarrow C, S')$ a minimal semi-resolution of $(f : X \rightarrow C, S)$.

The fundamental line bundle greatly influences the geometry of a minimal Weierstrass fibration. The line bundle \mathbb{L} has non-negative degree on C and is independent of choice of section S [Mir89]. Furthermore, \mathbb{L} determines the canonical bundle of X :

Proposition 2.8. [Mir89, Proposition III.1.1] *Let $(f : X \rightarrow C, S)$ be either (1) a Weierstrass fibration, or (2) a relatively minimal smooth elliptic surface. Then $\omega_X = f^*(\omega_C \otimes \mathbb{L})$.*

We prove a more general canonical bundle formula in [AB16a] (see Proposition 5.4).

Definition 2.9. We say that $f : X \rightarrow C$ is **properly elliptic** if $\deg(\omega_C \otimes \mathbb{L}) > 0$.

It is clear that X is properly elliptic if and only if the Kodaira dimension $\kappa(X) = 1$.

2.2. Singular fibers. When $(f : X \rightarrow C, S)$ is a smooth relatively minimal elliptic surface, then f has finitely many singular fibers. These are unions of rational curves with possibly non-reduced components whose dual graphs are *ADE* Dynkin diagrams. The possible singular fibers were classified independently by Kodaira and Néron.


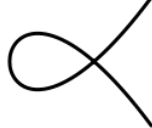

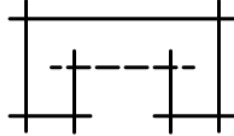
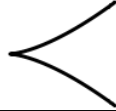

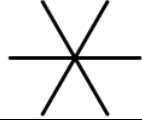

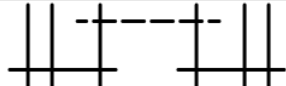
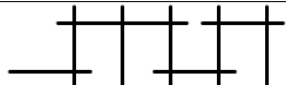

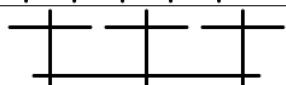
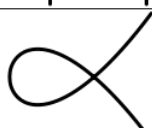
Table 1 gives the full classification in Kodaira's notation for the fiber. Fiber types I_n for $n \geq 1$ are reduced and normal crossings, fibers of type I_n^* , II^* , III^* , and IV^* are normal crossings but nonreduced, and fibers of type II , III and IV are reduced but not normal crossings.

For $f : X \rightarrow C$ isotrivial with $j = \infty$, La Nave classified the Weierstrass models with log canonical singularities in [LN, Lemma 3.2.2] (see also [AB16a, Section 5]). They have equation $y^2 = x^2(x - t^k)$ for $k = 0, 1$ and 2 and we call these N_0 , N_1 and N_2 fibers respectively.

Definition 2.10. A **reduced fiber** is the reduced divisor $F = (f^{-1}(p))^{red}$ underlying a (possibly nonreduced) scheme theoretic fiber $f^{-1}(p)$ for $p \in C$.

Remark 2.11. In the sequel we will consider pairs $(f : X \rightarrow C, S + \sum a_i F_i)$ where $(f : X \rightarrow C, S)$ is an elliptic surface with section and the F_i are reduced fibers. We will refer to the sum $\sum a_i F_i$ as the **reduced marked fibers**.

TABLE 1. Kodaira's classification of singular fibers of a smooth minimal elliptic surface

| Kodaira Type | # of components | Fiber |
|-------------------|-----------------------------|--|
| I_0 | 1 |  |
| I_1 | 1 (double pt) |  |
| I_2 | 2 (2 intersection pts) |  |
| $I_n, n \geq 2$ | n (n intersection pts) |  |
| II | 1 (cusp) |  |
| III | 2 (meet at double pt) |  |
| IV | 3 (meet at 1 pt) |  |
| I_0^* | 5 |  |
| $I_n^*, n \geq 1$ | $5 + n$ |  |
| II^* | 9 |  |
| III^* | 8 |  |
| IV^* | 7 |  |
| N_I | 1 (node) |  |

3. MODULI OF STABLE PAIRS

3.1. Stable pairs.

3.1.1. *The curve case.* First we review Hassett's weighted stable curves, as these will be used extensively, and they illuminate some of the basic geometric concepts.

Definition 3.1. Let $\mathcal{A} = (a_1, \dots, a_r)$ for $0 < a_i \leq 1$. An **\mathcal{A} -stable curve** is a pair $(C, D = \sum a_i p_i)$, of a reduced connected projective curve X together with a divisor D consisting of n weighted marked points p_i on C such that:

- C has at worst nodal singularities, the points p_i lie in the smooth locus of C , and for any subset $\{p_1, \dots, p_s\}$ with nonempty intersection we have $a_1 + \dots + a_s \leq 1$;
- $\omega_C(D)$ is ample.

In particular, if $\mathcal{A} = (1, \dots, 1)$, then one obtains an r -pointed stable curve in the sense of Knudsen [Knu83].

Theorem 3.2. [Has03] *Let $\mathcal{A} = (a_1, \dots, a_r)$ be a weight vector such that $0 < a_i \leq 1$. Suppose $g \geq 0$ is an integer. Then there is a smooth Deligne-Mumford stack $\overline{\mathcal{M}}_{g, \mathcal{A}}$ with projective coarse moduli space $\overline{M}_{g, \mathcal{A}}$ parametrizing \mathcal{A} -stable curves.*

Moreover, if one considers the domain of admissible weights, there is a *wall and chamber* decomposition – we say that $(a'_1, \dots, a'_r) \leq (a_1, \dots, a_r)$ if $a'_i \leq a_i$ for all i . Hassett proved the following theorem.

Theorem 3.3. [Has03] *There is a wall and chamber decomposition of the domain of admissible weights such that:*

- (1) *If \mathcal{A} and \mathcal{A}' are in the same chamber, then the moduli stacks and universal families are isomorphic.*
- (2) *If $\mathcal{A}' \leq \mathcal{A}$, then there is a reduction morphism $\overline{\mathcal{M}}_{g, \mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g, \mathcal{A}'}$ and a compatible contraction morphism on universal families.*

3.1.2. *Higher dimensions.* To compactify the moduli space of pairs of log general type in analogy to the above case of curves, one needs to introduce pairs on the boundary which have *semi-log canonical (slc) singularities*. We begin with their definition.

Definition 3.4. Let $(X, D = \sum d_i D_i)$ be a pair of a normal variety and a \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier. Suppose that there is a log resolution $f : Y \rightarrow X$ such that

$$K_Y + \sum a_E E = f^*(K_X + D),$$

where the sum goes over all irreducible divisors on Y . We say that the pair (X, D) has **log canonical singularities** (or is lc) if all $a_E \leq 1$.

Definition 3.5. Let (X, D) be a pair of a reduced variety and a \mathbb{Q} -divisor such that $K_X + D$ is \mathbb{Q} -Cartier. The pair (X, D) has **semi-log canonical singularities** (or is an slc pair) if:

- The variety X is S2,
- X has only double normal crossings in codimension 1, and
- If $\nu : X^\nu \rightarrow X$ is the normalization, then the pair $(X^\nu, \nu_*^{-1}D + D^\nu)$ is log canonical, where D^ν denotes the preimage of the double locus on X^ν .

Definition 3.6. A pair (X, D) of a projective variety and \mathbb{Q} -divisor is a **stable pair** if:

- (1) (X, D) is an slc pair, and
- (2) $\omega_X(D)$ is ample.

3.2. Families of stable pairs. In full generality, it has been difficult to construct a proper moduli space parametrizing stable pairs (X, D) with suitable numerical data. An example due to Hassett (see Section 1.2 in [KP15]), shows that when the coefficients of D are *not* all $> 1/2$, the divisor D might not deform as expected in a flat family of pure codimension 1 subvarieties of X – the limit of the divisor D may acquire an embedded point. However, we first make the following remarks:

Remark 3.7.

- Hassett and Alexeev (see [Has01] and [Ale08]) have demonstrated properness when all coefficients of D are equal to 1.
- It is well known to experts that, by a result of Kollár, the moduli space exists and is proper when the coefficients are all $> 1/2$ (see case 2 of Theorem 1.6.1 in [Ale15]).

While it is clear what the objects are (see Definition 3.6), it is not clear what the proper definition for families are, and thus it is unclear what exactly the moduli functor should be. Many functors have been suggested, but no functor seems to be “better” than any other. That being said, the projectivity results of [KP15], namely Theorem 1.1 in *loc. cit.*, is independent of the choice of functor, and applies to any moduli functor whose objects are stable pairs. We do remark that Kovács-Patakfalvi demonstrate their results using a proposed functor of Kollár (see Section 5 in [KP15]). We also note that it *is* clear what the definition of a stable family (i.e. a family of stable pairs) is over a *normal* base:

Definition 3.8. [KP15, Definition 2.11] A **family of stable pairs** of dimension n and volume v over a normal variety Y consists of a pair (X, D) and a flat proper surjective morphism $f : X \rightarrow Y$ such that

- (1) D avoids the generic and codimension 1 singular points of every fiber,
- (2) $K_{X/Y} + D$ is \mathbb{Q} -Cartier,
- (3) (X_y, D_y) is a connected n -dimensional stable pair for all $y \in Y$, and
- (4) $(K_{X_y} + D_y)^n = v$ for $y \in Y$.

We denote a family of stable pairs by $f : (X, D) \rightarrow Y$.

If we are satisfied working only over normal bases, then this definition of a family of stable pairs suffices. In fact, any moduli functor \mathcal{M} with

$$\mathcal{M}(Y) = \left\{ \begin{array}{l} \text{families of stable pairs } f : (X, D) \rightarrow Y \text{ of} \\ \text{dimension } n \text{ and volume } v \text{ as in Definition 3.8} \end{array} \right\}$$

for Y normal has the same normalization by Proposition A.7 (see also Definition 5.2 and Remark 5.15 of [KP15]). Therefore for many questions about moduli of stable pairs, one needs only consider families over a normal base.

4. WEIGHTED STABLE ELLIPTIC SURFACES

We are interested in the geometry of elliptic surface pairs $(f : X \rightarrow C, S + F)$ where $(f : X \rightarrow C, S)$ is an elliptic surface with section as above, and F is a boundary divisor corresponding to a finite sum of vertical fibers. Following the log minimal model program, we will study compactifications of the moduli space of irreducible elliptic surfaces with section and marked fibers obtained by allowing our surface pairs to degenerate to *semi-log canonical* (slc) pairs (see Definition 3.5). As such our surfaces can acquire non-normal singularities and break up into multiple components.

Let $\mathcal{A} \in \mathbb{Q}^r \cap [0, 1]^r$ be a rational *weight vector* with $0 \leq a_i \leq 1$. We consider elliptic surfaces marked by an \mathcal{A} -weighted sum $F_{\mathcal{A}} = \sum_{i=1}^r a_i F_i$ where F_i are reduced fibers. Note that the weights come with a natural partial ordering. We say that $\mathcal{A}' = (a'_1, \dots, a'_r) < \mathcal{A}$ if $a'_i \leq a_i$ for all i , and if the inequality is strict for at least one i . If $s \in \mathbb{Q}$ is a rational number, we write $\mathcal{A} \leq s$ ($\mathcal{A} \geq s$) if $a_i \leq s$ ($a_i \geq s$) for all i . Our goal is to compare moduli spaces for elliptic surfaces whose fibers have various weights \mathcal{A} .

As alluded to above, the boundary of our moduli space will parameterize slc pairs. The first definition we give, inspired by the minimal model program, yields a finite type and separated algebraic stack (see Theorem 4.4) with possibly too many components. In Definition 5.9, we will give a more refined definition of the objects that appear on the boundary of the compactified moduli stack when one runs stable reduction (see Theorem 8.6).

Definition 4.1. An \mathcal{A} -weighted slc elliptic surface with section $(f : X \rightarrow C, S + F_{\mathcal{A}})$, (see Figure 3) is an slc surface pair $(X, S + F_{\mathcal{A}})$ and a proper surjective morphism with connected fibers $f : X \rightarrow C$ to a projective nodal curve such that:

- (a) S is a section with generic points contained in the smooth locus of f , and $F_{\mathcal{A}} = \sum_i a_i F_i$ is an \mathcal{A} -weighted sum of reduced divisors contracted by f ,
- (b) every component of X is either an elliptic surface with section or a surface contracted to a point by f ,
- (c) any F_i that lies on an elliptic component is a reduced fiber (recall from Definition 2.10, that we mean the induced reduced structure) over a smooth point of C .

We say that $(f : X \rightarrow C, S + F_{\mathcal{A}})$ is an \mathcal{A} -stable elliptic surface if the \mathbb{Q} -Cartier divisor $K_X + S + F_{\mathcal{A}}$ is ample, that is, if $(X, S + F_{\mathcal{A}})$ is a stable pair.

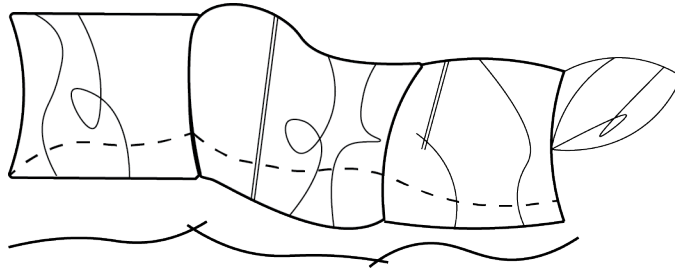


FIGURE 3. An \mathcal{A} -weighted slc elliptic surface.

We will elaborate on (b) of the above Definition 4.1 in Section 5.3. The components contracted to a point by f are precisely the *pseudoelliptic components* (see Definition 5.5) alluded to in the introduction. Once again, we will also refine the definition of the objects parametrized by the boundary in Definition 5.9.

4.1. Moduli functor for elliptic surfaces. Following [KP15], we introduce the following notion of a pseudofunctor (following Definition 5.2 of [KP15]) of stable elliptic surfaces:

Definition 4.2. Fix $v \in \mathbb{Q}_{>0}$. A pseudofunctor $\mathcal{E} : \mathfrak{Sch}_k \rightarrow \mathfrak{Grp}$ from the category of k -schemes to groupoids is a **moduli pseudofunctor for \mathcal{A} -stable elliptic surfaces of volume v** if for any normal variety T ,

$$\mathcal{E}(T) = \left\{ \left(\begin{array}{ccc} X & \xrightarrow{f} & C, S + F_{\mathcal{A}} \\ & \searrow g & \swarrow h \\ & T & \end{array} \right) \mid \begin{array}{l} (1) \ g : (X, S + F_{\mathcal{A}}) \rightarrow T \text{ is a flat family of} \\ \text{stable pairs of dimension 2 and volume} \\ \text{v as in Definition 3.8;} \\ (2) \ h \text{ is a flat family of connected nodal} \\ \text{curves;} \\ (3) \ f \text{ is a morphism over } T; \text{ and} \\ (4) \ \text{for each } t \in T, \text{ the fiber} \\ f_t : (X_t, S_t + (F_{\mathcal{A}})_t) \rightarrow C_t \text{ is an} \\ \mathcal{A}\text{-weighted slc elliptic surface with} \\ \text{section and reduced marked fibers.} \end{array} \right\}.$$

Let \mathcal{E}° be the subfunctor consisting of families with $(f_t : X_t \rightarrow C_t, S_t)$ an irreducible elliptic surface with section as in Definition 2.1. The **main component** \mathcal{E}^m will denote the closure $\overline{\mathcal{E}^\circ}$ in \mathcal{E} .

Remark 4.3. Despite the terminology, it is not true in general that \mathcal{E}^m is irreducible. Rather, it has components labeled by the configurations of singular fibers on the irreducible elliptic surfaces.

Theorem 4.4. *There exists a moduli pseudofunctor of \mathcal{A} -stable elliptic surfaces of volume v such that the main component \mathcal{E}^m is a separated Deligne-Mumford stack of finite type.*

Proof. In [KP15], a suitable pseudofunctor $\mathcal{M}_{v,I,n}$ for stable pairs (X, D) with volume v , coefficient set I and index n is defined. Here n is a fixed integer such that $n(K_X + D)$ is required to be Cartier. Furthermore, $\mathcal{M}_{v,I,n}$ is a finite type Deligne-Mumford stack with projective coarse space (see Proposition 5.11 and Corollary 6.3 in [KP15]). Take I to be the additively closed set generated by the weight vector \mathcal{A} . By boundedness for surface pairs (see Theorem 9.2. in [Ale94]), there exists an index n such that $n(K_X + S + F_{\mathcal{A}})$ is a very ample Cartier divisor for all \mathcal{A} -stable elliptic surfaces of volume v .

Consider the stack of stable pairs $\mathcal{M}_{v,I,n}$ and denote $\mathcal{M} := \mathcal{M}_{v,I,n}$ for convenience. Let $\mathcal{X} \rightarrow \mathcal{M}$ be the the universal family. Furthermore, let \mathfrak{M}_g be the algebraic stack of prestable curves with universal family $\mathfrak{C}_g \rightarrow \mathfrak{M}_g$. Consider the Hom-stack

$$\mathcal{H}om_{\mathcal{M} \times \mathfrak{M}_g}(\mathcal{X} \times \mathfrak{M}_g, \mathcal{M} \times \mathfrak{C}_g).$$

This is a quasi-separated algebraic stack locally of finite presentation with affine stabilizers by Theorem 1.2 in [HR14]. Now we consider the pseudofunctor given by

$$\mathcal{E}_{v,\mathcal{A},n} : B \mapsto \left\{ \begin{array}{ccc} (X, S + F_{\mathcal{A}}) & \xrightarrow{f} & C \\ & \searrow & \swarrow \\ & B & \end{array} \right\}$$

where $(X, S + F_{\mathcal{A}}) \rightarrow B$ is a flat family of stable pairs in the sense of [KP15], $C \rightarrow B$ is a flat family of pre-stable curves, and $(f_b : X_b \rightarrow C_b, S_b + (F_{\mathcal{A}})_b)$ is an \mathcal{A} -stable elliptic surface with volume v for each $b \in B$.

It is clear that $\mathcal{E}_{v,\mathcal{A},n}$ is a substack of the Hom-stack $\mathcal{H}om_{\mathcal{M} \times \mathfrak{M}_g}(\mathcal{X} \times \mathfrak{M}_g, \mathcal{M} \times \mathfrak{C}_g)$. The substack $\mathcal{E}_{v,\mathcal{A},n}^\circ$ parametrizing irreducible elliptic surfaces is an algebraic substack of the Hom-stack, as flatness and irreducibility are algebraic conditions. Thus the closure $\mathcal{E}_{v,\mathcal{A},n}^m$ in the Hom-stack is a quasi-separated algebraic stack locally of finite presentation with affine stabilizers, and is a pseudofunctor for \mathcal{A} -stable elliptic surfaces of volume v .

To prove that $\mathcal{E}_{v,\mathcal{A},n}^m$ is separated, let B be a smooth curve and let

$$(X^0, S^0 + F_{\mathcal{A}}^0) \xrightarrow{f^0} C^0 \longrightarrow B^0 = B \setminus p$$

be a flat family of \mathcal{A} -stable elliptic surfaces over the complement of a point $p \in B$. Suppose

$$\begin{aligned} (X, S + F_{\mathcal{A}}) &\xrightarrow{f} C \longrightarrow B \\ (X', S' + F'_{\mathcal{A}}) &\xrightarrow{f'} C' \longrightarrow B \end{aligned}$$

are two extensions to B .

Then $(X, S + F_{\mathcal{A}}) \rightarrow B$ and $(X', S' + F'_{\mathcal{A}}) \rightarrow B$ are two families of stable pairs over B with isomorphic restrictions to B^0 . Since log canonical models are unique, $(X', S' + F'_{\mathcal{A}}) = (X, S + F_{\mathcal{A}})$ over B . Furthermore, the compositions $S \rightarrow C$ and $S' \rightarrow C'$ are isomorphisms so $C \cong C'$ over B . Therefore, we have $f, f' : X \rightarrow C \rightarrow B$ with $f|_{X^0} = f'|_{X^0}$. Since $X \rightarrow B$ is flat, X^0 is dense in X , therefore $f = f'$ since C is separated. Thus an extension to B is unique and so $\mathcal{E}_{v,\mathcal{A},n}^m$ is separated.

Finally, we show that the stack is Deligne-Mumford, by showing that the objects have finitely many automorphisms. An automorphism of $(X, S + F_{\mathcal{A}}) \rightarrow C$ is an automorphism σ of the elliptic surface pair $(X, S + F_{\mathcal{A}})$, as well as an automorphism τ of C such that the automorphisms commute. Since the automorphism σ fixes the fibers $F_{\mathcal{A}}$, the compatibility of the automorphisms implies that τ actually fixes the marked points $f_* F_{\mathcal{A}}$ on C . We will show in Corollary 5.12 that the base curve is actually a weighted stable pointed curve in the sense of Hassett, and thus has finitely automorphisms. Moreover, there are finitely automorphisms of the stable surface pair (see e.g. [Iit82, 11.12]). \square

As it is not clear how to define families of stable pairs over a general base (see Remark 3.7), from now on we restrict to only considering families of elliptic surfaces over a normal base. Therefore define

$$\mathcal{E}_{v,\mathcal{A}} := (\mathcal{E}_{v,\mathcal{A},n}^m)^\nu$$

to be the normalization of the stack constructed in Theorem 4.4 (see Appendix A for a discussion on normalizations) and $\mathcal{U}_{v,\mathcal{A}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$ the pullback of the universal family. Then $\mathcal{E}_{v,\mathcal{A}}$ is a separated algebraic stack locally of finite type with affine stabilizers. By Proposition A.7, the stack $\mathcal{E}_{v,\mathcal{A}}$ is independent of n for n large enough, and more generally independent of the choice of pseudofunctor \mathcal{E} as in Definition 4.2.

5. LOG CANONICAL MODELS OF ELLIPTIC SURFACES

To study the compactification of the space of irreducible elliptic surface pairs by \mathcal{A} -stable elliptic surfaces, we need to understand the log canonical models of elliptic surface pairs and how they depend on the weights \mathcal{A} . That is, given $(f : X \rightarrow C, S + F_{\mathcal{A}})$, we need to know the stable models for all weights \mathcal{A} . This task was pursued for irreducible elliptic surfaces in [AB16a]. We summarize those results needed here and extend them to reducible \mathcal{A} -weighted slc elliptic surfaces.

5.1. Relative log canonical models. Our study of log canonical models of an elliptic surface pair $(f : X \rightarrow C, S + F_{\mathcal{A}})$ proceeds in two steps: first we compute the relative canonical model of $(X, S + F_{\mathcal{A}})$ over the curve C and then contract the section or whole components if necessary according to the log minimal model program. To this end, let $(f : X \rightarrow C, S + F_{\mathcal{A}})$ be an \mathcal{A} -weighted slc elliptic surface. We want to perform a sequence of extremal and log canonical contractions over C to make $K_X + S + F_{\mathcal{A}}$ an f -ample divisor.

Let $\nu : C' \rightarrow C$ be the normalization and let X' be the pullback:

$$\begin{array}{ccc} X' & \xrightarrow{\varphi} & X \\ f' \downarrow & & \downarrow f \\ C' & \xrightarrow{\nu} & C \end{array} .$$

Then $\varphi^*(K_X + S + F) = K_{X'} + G + S' + F'$ is f' -ample if and only if $K_X + S + F$ is f -ample. Here $\varphi^*S = S'$ is a section of f' and $F' = \varphi^*F$. The divisor G is the reduced divisor above the points of C' lying over the nodes of C . In particular, to compute the relative canonical model over C starting with a log smooth model, it suffices to assume C is smooth and $f : X \rightarrow C$ is an irreducible elliptic surface.

Furthermore, the question of whether $K_X + S + F$ is f -ample is local on the base. Therefore we may assume that $C = \text{Spec}(R)$ is a DVR with closed point s and generic point η . We then consider the log pair $(X, S + aF)$ where $F = f^*(s)^{red}$ and $0 \leq a \leq 1$.

Definition 5.1. Let $(f : X \rightarrow C, S + aF)$ be a relatively stable elliptic surface over the spectrum of a DVR. We say that X has a(n):

- (1) **twisted fiber** if the special fiber $f^*(s)$ is irreducible but non-reduced;

- (2) **intermediate fiber** if $f^*(s)$ is a nodal union of a rational component A and a possibly non-reduced arithmetic genus 1 component E such that the section meets $f^*(s)$ along the smooth locus of A .

Remark 5.2. The terminology for a twisted fiber comes from the fact that these fibers are exactly those that appear in the coarse space of a flat family $\mathcal{X} \rightarrow \mathcal{C}$ of *stable* elliptic curves over a *orbifold* base curve \mathcal{C} . Equivalently, a twisted fiber is obtained by taking the quotient of a family of stable curves over the spectrum of a DVR by a subgroup of the automorphism group of the central fiber. This notion is introduced in [AV97] for the purpose of obtaining fibered surfaces from twisted stable maps. See [AB16b] for more details for pairs.

Given an elliptic surface $f : X \rightarrow C$ over the spectrum of a DVR such that X has an intermediate fiber, we obtain the Weierstrass model of X by contracting the component E and we obtain the twisted model by contracting the component A . Thus the intermediate fiber can be seen as interpolating between the Weierstrass and twisted models (see Figure 4). The results of [AB16a] make this precise:

Theorem 5.3. *Let $(f : X \rightarrow C, S + aF)$ an elliptic surface pair over C the spectrum of a DVR with reduced special fiber F where either (a) F is one of the Kodaira singular fiber types, or (b) f is isotrivial with constant j -invariant ∞ .*

- (1) *If F is a type I_n or N_0 fiber, then the relative log canonical model is the Weierstrass model for all $0 \leq a \leq 1$.*
- (2) *For any other fiber type, there is an a_0 such that the relative log canonical model is*
 - (i) *the Weierstrass model for any $0 \leq a \leq a_0$,*
 - (ii) *a twisted fiber consisting of a single non-reduced component when $a = 1$, and*
 - (iii) *an intermediate fiber for any $a_0 < a < 1$.*

The constant $a_0 = 0$ for fibers of type I_n^, II^*, III^* and IV^* , and a_0 is as follows for the other fiber types:*

$$a_0 = \begin{cases} 5/6 & II \\ 3/4 & III \\ 2/3 & IV \\ 1/2 & N_1 \end{cases}$$

5.2. Canonical bundle formula. In [AB16a], we computed a formula for the canonical bundle.

Theorem 5.4. [AB16a, Theorem 1.2] *Let $f : X \rightarrow C$ be a fibration where X is an irreducible elliptic surface with section S , and let $F_{\mathcal{A}} = \sum a_i F_i$ be a sum of reduced marked fibers F_i with $0 \leq a_i \leq 1$. Suppose that $(X, S + F_{\mathcal{A}})$ is the relative log canonical model over C . Then*

$$\omega_X = f^*(\omega_C \otimes \mathbb{L}) \otimes \mathcal{O}_X(\Delta).$$

where Δ is effective and supported on fibers of type II, III, and IV contained in $\text{Supp}(F)$. The contribution of a type II, III or IV fiber to Δ is given by αE where E supports the unique

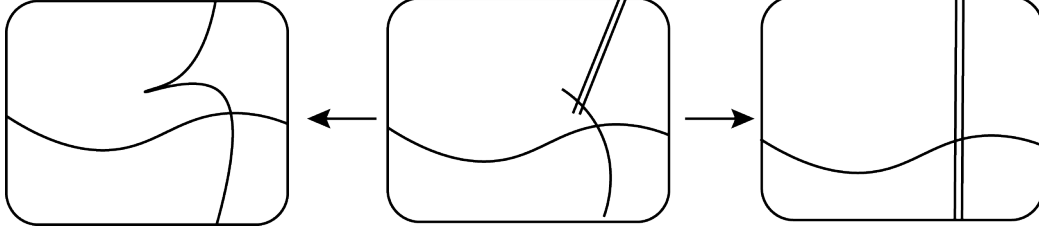


FIGURE 4. Here we illustrate the relative log canonical models and morphisms between them. From left to right: **Weierstrass** model ($0 \leq a \leq a_0$) – a single reduced and irreducible component meeting the section, **intermediate** model ($a_0 < a < 1$) – a nodal union of a reduced component meeting the section and a nonreduced component, and **twisted** model ($a = 1$) – a single nonreduced component meeting the section in a singular point of the surface.

nonreduced component of the fiber and

$$\alpha = \begin{cases} 4 & \text{II} \\ 2 & \text{III} \\ 1 & \text{IV} \end{cases}$$

It is important to emphasize here that only type II, III or IV fibers that are *not* in Weierstrass form affect the canonical bundle. If all of the type II, III and IV fibers of $f : X \rightarrow C$ are Weierstrass, then the usual canonical bundle formula $\omega_X = f^*(\omega_C \otimes \mathbb{L})$ holds.

5.3. Formation of pseudoelliptics. In [LN], La Nave studied compactifications of the moduli space of Weierstrass fibrations by stable elliptic surface pairs $(f : X \rightarrow C, S)$ – i.e. where $\mathcal{A} = 0$. There it was shown that the section of some irreducible components of a reducible elliptic surface may be contracted by the log MMP, inspiring the following.

Definition 5.5. A **pseudoelliptic surface** is a surface pair (Z, F) obtained by contracting the section of an irreducible elliptic surface pair $(f : X \rightarrow C, S + \bar{F})$. We call F the marked **pseudofibers** of Z . We call $(f : X \rightarrow C, S)$ the **associated elliptic surface** to (Z, F) .

There are two *types* of pseudoelliptic components that appear as irreducible components of a stable limit of elliptic surfaces.

Definition 5.6. Let $(T, 0)$ be a rooted tree with root vertex $0 \in V(T)$. We make $V(T)$ into a poset by declaring that $\alpha \leq \beta$ if vertex α lies on the unique minimal length path from vertex β to the root 0 . We denote by $T[i]$ the set of vertices of distance i from the root so that $T[0] = \{0\}$. Finally, if $\alpha \in T[i]$, we denote by $\alpha[1]$ the set of vertices $\beta \in T[i+1]$ with $\alpha \leq \beta$.

Definition 5.7. Let $(T, 0)$ be a rooted tree. A **pseudoelliptic tree** $(Y, F_{\mathcal{A}})$ with dual graph $(T, 0)$ is an slc pair consisting of the union of pseudoelliptic components Y_{α} with dual graph T constructed inductively: every component Y_{β} for $\beta \in \alpha[1]$ is attached to Y_{α} by gluing an

irreducible pseudofiber G_β of Y_β to the arithmetic genus 1 component E_α of an intermediate pseudofiber with reduced component A_α of Y_α . The \mathcal{A} -weighted marked fibers $F_{\mathcal{A}}$ satisfy

$$\text{Coeff}(A_\alpha, F_{\mathcal{A}}) = \sum_{\beta \in \alpha[1]} \sum_{D \in \text{Supp}(F_{\mathcal{A}}|_{Y_\beta})} \text{Coeff}(D, F_{\mathcal{A}}).$$

A component $(Y_\alpha, F_{\mathcal{A}}|_{Y_\alpha})$ is a **Type I** pseudoelliptic (See Figure 5).

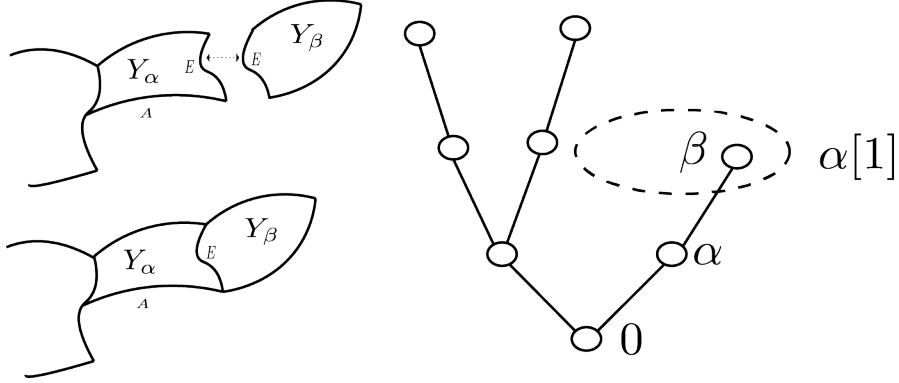


FIGURE 5. A pseudoelliptic tree is constructed inductively by attaching a *Type I* pseudoelliptic surface Y_β to Y_α for each $\beta \in \alpha[1]$ as pictured. The component A on Y_α is marked by the sum of the weights of the markings on Y_β .

Definition 5.8. A pseudoelliptic surface of **Type II** (see Figure 6) is formed by the log canonical contraction of a section of an elliptic component attached along twisted or stable fibers.

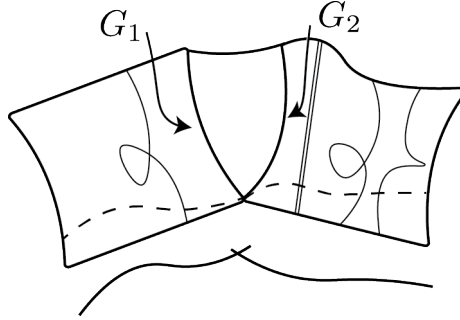


FIGURE 6. A pseudoelliptic surface of *Type II* with attaching fibers G_1 and G_2 .

One new insight is that the section S is often contracted even for \mathcal{A} -weighted elliptic surfaces with small but nonzero weights. In fact, we will see (Section 5.4) that contracting the section of a component to form a pseudoelliptic corresponds to stabilizing the base curve as an \mathcal{A} -stable curve in the sense of Hassett (see Section 3.1.1).

We can now single out the particular slc elliptic surfaces that will appear on the boundary of the main components of the moduli space (see Figure 7).

Definition 5.9. An \mathcal{A} -**broken elliptic surface** is an \mathcal{A} -weighted slc elliptic surface pair $(f : X \rightarrow C, S + F_{\mathcal{A}})$ such that (see Figure 7)

- (a) each component of X contracted by f is a type I or type II pseudoelliptic surface with marked pseudofibers,
- (b) the elliptic components and type II pseudoelliptics are attached along irreducible fibers,
- (c) the type I pseudoelliptics appear in pseudoelliptic trees attached by gluing an irreducible pseudofiber G_0 on the root to an arithmetic genus 1 component E of an intermediate (pseudo)fiber of an elliptic (type II pseudoelliptic) component.

We say $(f : X \rightarrow C, S + F_{\mathcal{A}})$ is an \mathcal{A} -**broken stable elliptic surface** if $(X, S + F_{\mathcal{A}})$ is a stable pair.

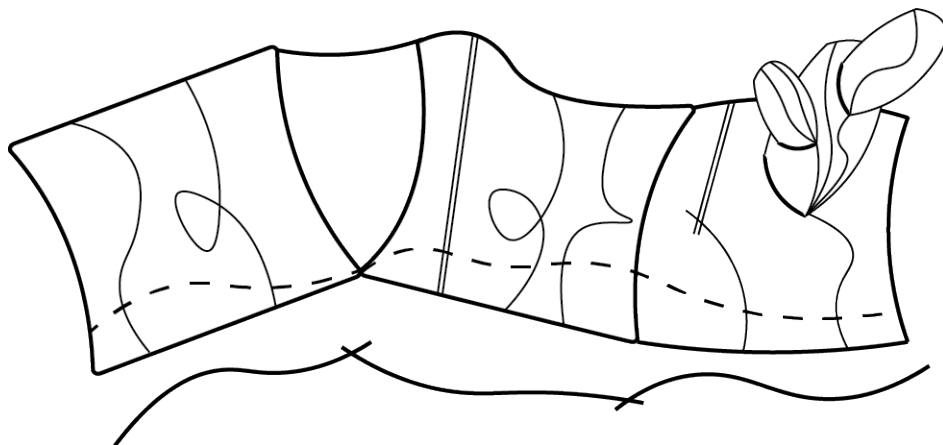


FIGURE 7. An \mathcal{A} -weighted broken elliptic surface.

Remark 5.10. For each pseudoelliptic component $X_0 \subset X$ with associated elliptic surface $f_0 : Y_0 \rightarrow C_0$ and morphism $\mu_0 : Y_0 \rightarrow X_0$ contracting the section, there is a unique slc elliptic surface $f' : X' \rightarrow C'$ with $Y_0 \subset X'$ and $f'|_{Y_0} = f_0$. There is a morphism $\mu : X' \rightarrow X$ contracting the section of Y_0 . The surface X' is a broken elliptic surface if and only if X_0 is a type II pseudoelliptic.

5.4. Pseudoelliptic contractions. In this subsection, we record various statements about the formation and contraction of pseudoelliptic components by the log canonical linear series. Some results are mild generalizations of statements in [AB16a] for irreducible elliptic surface pairs.

The following describes how the log canonical divisor class intersects the section (see also Proposition 6.5 in [AB16a] and Proposition 4.3.2 in [LN]). This determines when the section of a component contracts to form a pseudoelliptic surface.

Given an \mathcal{A} -broken elliptic surface $(f : X \rightarrow C, S + F_{\mathcal{A}})$, we obtain an \mathcal{A} -weighted pointed curve $(C, f_*F_{\mathcal{A}})$. We form the dual graph of C by assigning a vertex to each irreducible

component $C_\alpha \subset C$ and an edge for each node. Let v_α be the valence of C_α in the dual graph and $g(C_\alpha)$ the geometric genus of C_α .

Proposition 5.11. [AB16b, Proposition 5.3] *Let $(f : X \rightarrow C, S + F_{\mathcal{A}})$ be an \mathcal{A} -broken elliptic surface. Let $(C, f_*F_{\mathcal{A}})$ be the \mathcal{A} -weighted pointed curve and $C \subset C_\alpha$ an irreducible component. Then for the component S_α of the section lying above C_α , we have*

$$\begin{aligned} (K_X + S + F) \cdot S_\alpha &= 2g(C_\alpha) - 2 + v_\alpha + \deg(f_*F_{\mathcal{A}}|_{C_\alpha}) \\ &= \deg(\omega_C(f_*F_{\mathcal{A}})|_{C_\alpha}). \end{aligned}$$

Proof. The case where $\mathcal{A} = 1$ is precisely Proposition 5.3 of [AB16b] (see also Proposition 6.5 in [AB16a] and Proposition 4.3.2 in [LN]). This more general case follows from the adjunction formula, as the section passes through the smooth locus of the surface in a neighborhood of any fiber that is not marked with coefficient $a_i = 1$. \square

Corollary 5.12. [AB16a, Corollary 6.7 & 6.8] *Let $(f : X \rightarrow C, S + F_{\mathcal{A}})$ be the relative log canonical model over C with f flat. Then $(K_X + S + F_{\mathcal{A}}) \cdot S_\alpha > 0$ for every component S_α of S if and only if $(C, f_*F_{\mathcal{A}})$ is an \mathcal{A} -pointed stable curve. In this case, the relative log canonical model over C is stable.*

Corollary 5.13. [AB16a, Corollary 6.9] *The log minimal model program contracts the section of an elliptic component $X_\alpha \rightarrow C_\alpha$ of $(f : X \rightarrow C, S + F_{\mathcal{A}})$ to produce a pseudoelliptic if and only if either:*

- (a) $C \cong \mathbb{P}^1$ and $\sum a_i \leq 2$, or
- (b) C is a genus 1 curve and $a_i = 0$ for all i .

Proposition 5.14. [AB16a, Proposition 7.1] *Let $f : X \rightarrow C$ be an irreducible properly elliptic surface with section S . Then $K_X + S$ is big.*

We will define \mathbb{L} on \mathbb{P}^1 for a pseudoelliptic surface X to be the fundamental line bundle (see Definition 2.7) for $\pi : Y \rightarrow \mathbb{P}^1$, where Y is the corresponding elliptic surface.

Proposition 5.15. [AB16a, Proposition 7.4] *Let $(X, F_{\mathcal{A}})$ be an \mathcal{A} -weighted slc pseudoelliptic surface with marked pseudofibers $F_{\mathcal{A}}$. Denote by Y the corresponding elliptic surface and $\mu : Y \rightarrow X$ the contraction of the section. Suppose $\deg \mathbb{L} = 1$ and $0 \leq \mathcal{A} \leq 1$ such that $K_X + F_{\mathcal{A}}$ is a nef and \mathbb{Q} -Cartier \mathbb{Q} -divisor. Then either*

- i) $K_X + F_{\mathcal{A}}$ is big and the log canonical model is an elliptic or pseudoelliptic surface;
- ii) $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} \mu_*\Sigma$ where Σ is a multisection of Y and the log canonical contraction maps X onto a rational curve; or
- iii) $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ and the log canonical map contracts X to a point.

The cases above correspond to $K_X + G + F_{\mathcal{A}}$ having Iitaka dimension 2, 1 and 0 respectively.

Remark 5.16. The proof of [AB16a, Proposition 7.4] actually gives a method for determining which situation of (i), (ii), and (iii) we are in. Indeed since $K_X + F_{\mathcal{A}}$ is nef, it is big if and only if $(K_X + F_{\mathcal{A}})^2 > 0$. Furthermore, $K_X + F_{\mathcal{A}} \sim_{\mathbb{Q}} 0$ if and only if $t = 0$ where

$$K_Y + tS + \tilde{F}_{\mathcal{A}} = \mu^*(K_X + F_{\mathcal{A}}).$$

So if $K_X + G + F_A$ is not big, it suffices to compute whether $t > 0$ or $t = 0$ to decide if the log canonical map contracts the pseudoelliptic to a curve or to a point.

Proposition 5.17. *Let $(f : X \rightarrow \mathbb{P}^1, S + F_B)$ be an irreducible \mathcal{B} -stable elliptic surface such that $\deg \mathbb{L} = 1$. Suppose $\mathcal{B} = (1, b_2, \dots, b_s)$, $0 < \mathcal{A} \leq \mathcal{B}$ such that $a_1 = b_1 = 1$, and F_1 is a type I_n fiber. Then $K_X + S + F_A$ is big. In particular, this holds if X is a component of an slc elliptic surface attached along a stable fiber.*

Proof. By assumption, $K_X = -G + \Delta$ for G a fiber. All fibers are linearly equivalent as C is rational, and type I_n fibers are reduced so that $F_1 \sim_{\mathbb{Q}} G$. Thus $K_X + F_1 = \Delta$ is effective and

$$K_X + S + F_B = \Delta + S + \sum_{i=2}^s b_i F_i,$$

$$K_X + S + F_A = \Delta + S + \sum_{i=2}^s a_i F_i$$

with $0 < a_i \leq b_i$ and $K_X + S + F_B$ ample. Since $a_i > 0$, for m large enough we can write $m(K_X + S + F_A) - (K_X + S + F_B) = D$ where D is effective. Therefore $K_X + S + F_A$ is big by Kodaira's lemma. \square

Proposition 5.18. [AB16a, Proposition 7.3] *Let $(f : X \rightarrow \mathbb{P}^1, S + F_A)$ be an irreducible \mathcal{A} -slc elliptic surface with section and marked fibers and suppose that $\deg \mathbb{L} = 2$.*

- (a) *If $\mathcal{A} > 0$, then $K_X + S + F_A$ is big and the log canonical model is either the relative log canonical model, or the pseudoelliptic obtained by contracting the section of the relative log canonical model.*
- (b) *If $\mathcal{A} = 0$, then the minimal model program results in a pseudoelliptic surface with a log canonical contraction that contracts this surface to a point.*

Proposition 5.19. *Let $(X, G_1 + G_2)$ be a Type II pseudoelliptic surface (see Definition 5.8) with attaching fibers G_1 and G_2 . Then $K_X + G_1 + G_2$ is big.*

Proof. Consider the blowup $\mu : Z \rightarrow X$, where $(Z \rightarrow \mathbb{P}^1, S + G_1 + G_2)$ is the corresponding elliptic surface. Note that we abuse notation and call G_1 and G_2 the strict transform of the pseudofibers through the blowup. Taking the relative log canonical model, we obtain a pair $(Y \rightarrow \mathbb{P}^1, S + G_1 + G_2)$, where by construction $K_Y + S + G_1 + G_2$ is relatively ample. Note that $(K_Y + S + G_1 + G_2) \cdot S = 0$ by Proposition 5.11 and $K_Y + S + G_1 + G_2$ has positive degree on all other curve classes as it is f -ample. Therefore $K_Y + S + G_1 + G_2$ is actually nef, and thus semiample by Proposition 6.2. Therefore the only curve contracted by $|m(K_Y + S + G_1 + G_2)|$ is the section S and the log canonical model of $(X, G_1 + G_2)$ is the corresponding pseudoelliptic surface of $(f : Y \rightarrow \mathbb{P}^1, S + G_1 + G_2)$. Therefore, $(X, G_1 + G_2)$ is log general type and $K_X + G_1 + G_2$ must be big. \square

6. PRELIMINARIES FROM THE LOG MINIMAL MODEL PROGRAM (MMP)

We work with \mathbb{Q} -divisors. Whenever we write equality for divisors, e.g. $K_X = \Delta$, unless otherwise noted, we mean \mathbb{Q} -linear equivalence. We first state a lemma.

Lemma 6.1. [KM98, 2.35] *If $(X, D + D')$ is an lc pair, and D' is an effective \mathbb{Q} -divisor, then (X, D) is also an lc pair.*

We will make repeated use of abundance for slc surface pairs:

Proposition 6.2 (Abundance for slc surfaces, see [AFKM02] and [Kaw92]). *Let (X, D) be an slc surface pair. If $K_X + D$ is nef, then it is semiample.*

The following results are standard (see for example [AB16a, Section 3]).

Lemma 6.3. *Let X be seminormal and $\mu : Y \rightarrow X$ a projective morphism with connected fibers. Then for any coherent sheaf \mathcal{F} on X , we have that $\mu_*\mu^*\mathcal{F} = \mathcal{F}$.*

Proposition 6.4. *Let (X, Δ) be an slc pair and $\mu : Y \rightarrow X$ a (partial) semi-resolution. Write*

$$K_Y + \mu_*^{-1}\Delta + \Gamma = \mu^*(K_X + \Delta) + B$$

where $\Gamma = \sum_i E_i$ is the exceptional divisor of μ and B is effective and exceptional. Then

$$\mu_*\mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma)) \cong \mathcal{O}_X(m(K_X + \Delta)).$$

Corollary 6.5. *Notation as above; the morphism μ induces an isomorphism of global sections*

$$H^0(X, \mathcal{O}_X(m(K_X + \Delta))) \cong H^0(Y, \mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma))).$$

In particular, $K_X + \Delta$ is big if and only if $K_Y + \mu_^{-1}\Delta + \Gamma$ is big.*

Proof. The first part is the definition of pushforwards. The second statement follows since $\dim Y = \dim X$. \square

Corollary 6.6. *Notation as above; the morphism μ induces an injection*

$$H^i(X, \mathcal{O}_X(m(K_X + \Delta))) \hookrightarrow H^i(Y, \mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma)))$$

for all $i > 0$.

Proof. This follows from the Leray spectral sequence for μ applied to $\mathcal{O}_Y(m(K_Y + \mu_*^{-1}\Delta + \Gamma))$. \square

Proposition 6.7. *Let X be a smooth projective surface and D a divisor on X such that $H^2(X, \mathcal{O}_X(D)) = 0$. If $D^2 > 0$, then D is big.*

Proof. Applying Riemann-Roch for surfaces and using that $h^2(X, \mathcal{O}_X(D)) = 0$, we get that

$$h^0(X, \mathcal{O}_X(mD)) \geq \frac{1}{2} (m^2 D^2 - mD.K_X) + \chi(\mathcal{O}_X)$$

so D is big by definition. \square

6.1. Vanishing theorems. The existence of morphisms between the moduli spaces will rely on the proof of a vanishing theorem for higher cohomologies which implies invariance of log plurigenera for a family of \mathcal{A} -weighted broken elliptic surfaces (see Section 7). There are various preliminary vanishing results we will use along the way that we record here for convenience.

The first is a version of Grauert-Riemenschneider vanishing theorem for surfaces. The proof is analagous to the proof of [Kol13, Theorem 10.4].

Proposition 6.8. *Let X be an slc surface and $f : X \rightarrow Y$ a proper, generically finite morphism with exceptional curves C_i such that $E = \bigcup_i C_i$ is a connected curve with arithmetic genus 0. Let L be a line bundle on X . Suppose*

- (1) *each C_i is a \mathbb{Q} -Cartier divisor;*
- (2) *$C_i \cdot E \leq 0$ for all i ; and*
- (3) *$\deg(L|_{C_i}) = 0$ for all i .*

Then $R^1 f_ L = 0$.*

Proof. Let $Z = \sum_{i=1}^s r_i C_i$ be an effective integral cycle. Then we prove, using induction, that the stalk $(R^1 f_* L)_{Y,p} = \varprojlim_Z H^1(Z, L|_Z) = 0$. As f is finite away from $p = f(E)$, this gives $R^1 f_* L = 0$.

Let C_i be an irreducible curve contained in $\text{Supp}(Z)$, and let $Z_i = Z - C_i$. Consider the short exact sequence:

$$0 \rightarrow \mathcal{O}_{C_i} \otimes \mathcal{O}_X(-Z_i) \rightarrow \mathcal{O}_Z \rightarrow \mathcal{O}_{Z_i} \rightarrow 0.$$

Tensoring with L we obtain:

$$0 \rightarrow \mathcal{O}_{C_i} \otimes L(-Z_i) \rightarrow L \otimes \mathcal{O}_Z \rightarrow L \otimes \mathcal{O}_{Z_i} \rightarrow 0.$$

By induction on $\sum r_i$, we know that $H^1(Z_i, L|_{Z_i}) = 0$. Therefore, it suffices to show that $H^1(C_i, \mathcal{O}_{C_i} \otimes L(-Z_i)) = 0$ for some i . Moreover, by Serre duality it suffices to show that

$$L \cdot C_i - Z_i \cdot C_i > \deg \omega_{C_i} = -2.$$

By assumption, $L \cdot C_i = 0$, so it suffices to show that $-Z_i \cdot C_i > -2$, or equivalently that $Z_i \cdot C_i < 2$. This follows from Artin's results on intersection theory of exceptional curves for rational surface singularities [Art66] applied to the normalization of X , as C_i and E are rational exceptional curves. \square

Next we will use Fujino's generalization of the Kawamata-Viehweg vanishing theorem for slc pairs. Before stating the result, we will need to make a preliminary definition.

Definition 6.9. Let (X, Δ) be a semi-log canonical pair and let $\nu : X^\nu \rightarrow X$ be the normalization. Let Θ be a divisor on X^ν , so that $(K_{X^\nu} + \Theta) = \nu^*(K_X + \Delta)$. A subvariety $W \subset X$ is called an **slc center** of (X, Δ) if there exists a resolution of singularities $f : Y \rightarrow X^\nu$ and a prime divisor E on Y such that the discrepancies $a(E, X^\nu, \Theta) = -1$ and $\nu \circ f(E) = W$. A subvariety $W \subset X$ is called an **slc stratum** if W is an slc center, or an irreducible component.

Now we state Fujino's theorem.

Theorem 6.10. [Fuj14, Theorem 1.10] *Let (X, Δ) be a projective semi-log canonical pair, L a \mathbb{Q} -Cartier divisor whose support does not contain any irreducible components of the conductor, and $f : X \rightarrow S$ a projective morphism. Suppose $L - (K_X + \Delta)$ is f -nef and additionally is f -big over each slc stratum of (X, Δ) . Then $R^i f_* \mathcal{O}_X(L) = 0$ for $i > 0$.*

Proposition 6.11. (i) *Let $(f : X \rightarrow C, S + F_{\mathcal{A}})$ be an irreducible slc elliptic surface such that $K_X + S + F_{\mathcal{A}}$ is f -nef and $(X, S + F_{\mathcal{B}})$ is stable over C for some $0 \leq \mathcal{A} \leq \mathcal{B} \leq 1$. Then*

$$H^2(X, \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$$

for any $m \geq 2$.

(ii) *Let $(X, F_{\mathcal{A}})$ be an irreducible pseudoelliptic surface such that $K_X + F_{\mathcal{A}}$ is nef and $(X, F_{\mathcal{B}})$ is stable for some $\mathcal{A} \leq \mathcal{B} \leq 1$. Then*

$$H^2(X, \mathcal{O}_X(m(K_X + F_{\mathcal{A}}))) = 0$$

for any $m \geq 2$.

Proof. First we prove (i) using Theorem 6.10. For a generic fiber G of f , we have the equality $(K_X + S + F_{\mathcal{A}}).G = S.G = 1$ since $K_X + F_{\mathcal{A}}$ is supported on fiber components. It follows that $K_X + S + F_{\mathcal{A}}$ is f -big on X . The one dimensional slc strata are the section S and the components of $\lfloor F_{\mathcal{A}} \rfloor$, the fibers marked with coefficient 1.

Any fiber $G \in \text{Supp}(\lfloor F_{\mathcal{A}} \rfloor)$ must be the relative log canonical model for coefficient 1 and therefore is an irreducible fiber (see Section 5). Thus $(K_X + S + F_{\mathcal{A}}).G = S.G = 1$ so $K_X + S + F_{\mathcal{A}}$ is f -big on each component of $\lfloor F_{\mathcal{A}} \rfloor$. Finally $K_X + S + F_{\mathcal{A}}$ is clearly f -big on S and on the zero dimensional slc strata.

By Theorem 6.10, we have the vanishing

$$R^i f_* \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}})) = 0,$$

for $i > 0$ and $m \geq 2$. Therefore,

$$H^0(C, R^2 f_* \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = H^1(C, R^1 f_* \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0.$$

Since $H^2(C, f_* \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$ on a curve, we obtain that

$$H^2(X, \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$$

by the Leray spectral sequence.

For (ii), we may take a partial log resolution $\mu : Y \rightarrow X$ by the associated elliptic surface so that $(f : Y \rightarrow C, S + \mu_*^{-1} F_{\mathcal{A}})$ is an \mathcal{A} -weighted slc elliptic surface. Corollary 6.6 and the projection formula reduce to the situation in (i). \square

7. INVARIANCE OF LOG PLURIGENERA FOR BROKEN ELLIPTIC SURFACES

In [AB16b], we investigated the stable pairs compactification of the space of *twisted elliptic surfaces* using the theory of twisted stable maps. A twisted elliptic surface is an irreducible \mathcal{A} -stable elliptic surface $(f : X \rightarrow C, S + F_{\mathcal{A}})$ for the constant weight vector $\mathcal{A} = (1, \dots, 1)$ satisfying the property that the support of every non-stable fiber is contained in $\text{Supp}(F_{\mathcal{A}})$. In particular, the compactification of the space of twisted elliptic surfaces is a component of the space $\mathcal{E}_{v, \mathcal{A}}$ we denote $\mathcal{F}_v(1, 1)$.

The main result of [AB16b] regarding the space $\mathcal{F}_v(1, 1)$ is the characterization of the boundary components as consisting of broken elliptic surfaces (see Theorem 1.4 [AB16b]). Our goal is to generalize this result to \mathcal{A} -stable elliptic surfaces for arbitrary weight and use it to construct various morphisms between the moduli spaces for different weights analagous to the reduction morphisms of Hassett spaces (see Theorem 3.3).

To show the existence of such morphisms on the level of moduli spaces and universal families, we demonstrate that the pushforwards of the pluri-log canonical sheaves of a family are compatible with base change so that the construction of log canonical models is functorial in families. The main technical step is the following vanishing theorem for the pluri-log canonical divisor which we prove in this section:

Theorem 7.1. *Let $(f : X \rightarrow C, S + F_{\mathcal{B}})$ be a \mathcal{B} -broken stable elliptic surface with section S and reduced marked fibers $F_{\mathcal{B}}$. Let $0 \leq \mathcal{A} \leq \mathcal{B}$ such that $K_X + S + F_{\mathcal{A}}$ is nef and \mathbb{Q} -Cartier. Suppose further that either (a) $p_g(C) \neq 1$, or (b) \mathcal{A} is not identically zero. Then*

$$H^i\left(X, \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))\right) = 0$$

for $i > 0$ and $m \geq 2$ divisible enough.

Remark 7.2.

- (1) Let $\mathcal{A}_t = t\mathcal{B} + (1 - t)\mathcal{A}$. Since the nef cone is the closure of the ample cone, the divisor $K_X + S + F_{\mathcal{A}_t}$ is also ample for $t > 0$. That is, $K_X + S + F_{\mathcal{A}}$ is the first time we drop from ample to nef along the segment connecting \mathcal{B} to \mathcal{A} .
- (2) We consider the case $p_g(C) = 1$ and $\mathcal{A} = 0$ in Theorem 7.8.

Proof. For convenience we sometimes denote the \mathbb{Q} -line bundle $L^{[m]} := \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))$. The proof will proceed through several steps.

Step 1: First we carefully break X up into several components.

Let $Y \subset X$ be a union of irreducible components and let X' be a union of the complementary irreducible components. Then there is an exact sequence

$$0 \rightarrow L^{[m]}|_{X'}(-M) \rightarrow L^{[m]} \rightarrow L^{[m]}|_Y \rightarrow 0$$

where $M = \sum_{j=1}^s M_j$ is the sum of fiber components along which X' and Y are attached to obtain X (see the proof of [KK02, Corollary 10.34]). Since $\mathcal{O}_{X'}(K_X|_{X'}) = \mathcal{O}_{X'}(K_{X'} + M)$

and $\mathcal{O}_Y(K_X|_Y) = \mathcal{O}_Y(K_Y + M)$, we see that

$$\begin{aligned} L^{[m]}|_Y &= \mathcal{O}_Y\left(m(K_Y + S|_Y + F_{\mathcal{A}}|_Y + M)\right) \\ L^{[m]}|_{X'}(-M) &= \mathcal{O}_{X'}\left(m(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M)\right). \end{aligned}$$

By the long exact sequence of cohomology, it suffices to prove vanishing for the divisor $L^{[m]}|_{X'}(-M)$ on X' and $L^{[m]}|_Y$ on Y . To do this, we need to guarantee some positivity for $L^{[m]}|_{X'}(-M)$, namely that it is nef. This is not immediate due to the twisting by $-M$, and therefore we need to pick X' and Y judiciously to ensure that twisting by $-M$ still yields a nef divisor. Note on the other hand that $L^{[m]}|_Y$ is automatically nef.

Let Y be a pseudoelliptic tree (see Definition 5.7) indexed by the rooted tree $(T, 0)$ with root component Y_0 . Suppose that Y is attached to X' by gluing an irreducible pseudofiber of Y_0 to the arithmetic genus 1 component of an intermediate fiber on X' . In this case M is an irreducible curve. Let A denote the other component of the intermediate fiber of X' . Suppose finally that $\text{Coeff}(A, F_{\mathcal{A}}) < \text{Coeff}(A, F_{\mathcal{B}})$.

Lemma 7.3. *In the situation above, $L^{[m]}|_{X'}(-M)$ and $L^{[m]}|_Y$ are nef and \mathbb{Q} -Cartier.*

Proof. $L^{[m]}|_Y$ is nef and \mathbb{Q} -Cartier as it is the restriction of a nef and \mathbb{Q} -Cartier divisor. On the other hand, we need to check that

$$L^{[m]}|_{X'}(-M) = \mathcal{O}_{X'}\left(m(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M)\right)$$

is nef and \mathbb{Q} -Cartier on X' . For \mathbb{Q} -Cartier, it suffices to note that X' has quotient singularities in a neighborhood of M (see Section 6.2 of [AB16b]). To see that it is nef, note that we only need to check

$$\begin{aligned} \left(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M\right) \cdot M &\geq 0 \\ \left(K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + \frac{m-1}{m}M\right) \cdot A &\geq 0 \end{aligned}$$

since $K_{X'} + S|_{X'} + F_{\mathcal{A}}|_{X'} + M$ is nef and reducing the coefficient of M does not affect the degree on the other components of the marked divisor. Furthermore, the intersections we are computing are all on the single component of X' containing A , so we may suppose X' is irreducible.

The first inequality is clear – recall that $M^2 < 0$, so reducing its coefficient *increases* the intersection with M . For the second inequality, we take a log resolution $\mu : X_0 \rightarrow X'$ if necessary, so that we can assume that A lies on an elliptic component $f_0 : X_0 \rightarrow C_0$ with section S_0 . Using the fact that the \mathcal{B} -weighted divisor $K_X + S + F_{\mathcal{B}}$ is ample, we see that $K_{X_0} + S_0 + F_{\mathcal{B}}|_{X_0} + M$ is f_0 -ample. Furthermore A is disjoint from the other marked fibers and $A^2 < 0$, so that decreasing the coefficient of A *increases* the degree on A . That is,

$$(K_{X_0} + S_0 + F_{\mathcal{A}}|_{X_0} + M) \cdot A > 0$$

so for large enough m ,

$$\left(K_{X_0} + S_0 + F_{\mathcal{A}}|_{X_0} + \frac{m-1}{m}M\right) \cdot A > 0.$$

In particular, $K_{X_0} + S_0 + F_{\mathcal{A}}|_{X_0} + \frac{m-1}{m}M$ is f_0 -nef. Thus, after possibly contracting the section if necessary, we obtain a log minimal model $(X', \mu_*(S_0 + F_{\mathcal{A}}|_{X_0} + \frac{m-1}{m}M))$. In particular, $K_{X'} + (S + F_{\mathcal{A}})|_{X'} + \frac{m-1}{m}M$ is nef. \square

Now we check that the condition $\text{Coeff}(A, F_{\mathcal{A}}) < \text{Coeff}(A, F_{\mathcal{B}})$ is satisfied whenever Y is a pseudoelliptic tree which contains at least one marked divisor whose coefficient is lowered. Indeed, if Y_{α} is a component and A_{α} is the reduced component of an intermediate fiber where another pseudoelliptic Y_{β} with $\beta \geq \alpha$ is attached, then

$$\text{Coeff}(A_{\alpha}, F_{\mathcal{A}}) = \sum_{D \subset \text{Supp}(F_{\mathcal{A}}|_{Y_{\beta}})} \text{Coeff}(D, F_{\mathcal{A}})$$

is a sum of the coefficients of marked fibers on Y_{β} . In particular, if A as above is the reduced component of an intermediate fiber on X' where the root component Y_0 of Y is attached, then $\text{Coeff}(A, F_{\mathcal{A}}) < \text{Coeff}(A, F_{\mathcal{B}})$ since there is some D on some Y_{β} with $\text{Coeff}(D, F_{\mathcal{A}}) < \text{Coeff}(D, F_{\mathcal{B}})$.

Now by induction on the number of pseudoelliptic trees where we have reduced coefficients, we use the long exact sequence on cohomology associated to

$$0 \rightarrow L^{[m]}|_{X'}(-M) \rightarrow L^{[m]} \rightarrow L^{[m]}|_Y \rightarrow 0$$

and reduce to proving vanishing for the following two cases:

- (1) $(X, S + F_{\mathcal{A}})$ is an slc \mathcal{A} -broken elliptic surface such that $F_{\mathcal{A}}|_Y = F_{\mathcal{B}}|_Y$ for any pseudoelliptic tree, or
- (2) $(X, F_{\mathcal{A}})$ is an slc pseudoelliptic tree.

We will denote this pair (X, Δ) and take care to note which case we are in if necessary.

Step 2: We consider Case 1. Here we show that we may assume that $K_X + S + F_{\mathcal{A}}$ is big on every component of X . Indeed $K_X + S + F_{\mathcal{A}}$ is *ample* on every pseudoelliptic of Type I by assumption. By Proposition 5.19, it is big on every pseudoelliptic of Type II (see Definition 5.8) and every elliptic component with $\deg \mathbb{L} > 0$.

We are left to consider a component $X_1 \cong E \times C_1$ isomorphic to a product with section S_1 . By Proposition 5.11, if $(K_X + S + F_{\mathcal{A}})|_{X_1}$ is nef but not big, then C_1 is rational and $(K_X + S + F_{\mathcal{A}}) \cdot S_1 = 0$. In this case, the log canonical morphism factors through a morphism $\mu : X \rightarrow Z$ which contracts the component X_1 onto E and is an isomorphism away from X_1 .

Now $(Z, \mu_*(S + F_{\mathcal{A}}))$ is an \mathcal{A} -broken elliptic surface and

$$\mu_*\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}})) = \mathcal{O}_Z\left(m(K_Z + \mu_*(S + F_{\mathcal{A}}))\right).$$

Therefore we want to show $R^i \mu_* L^{[m]} = 0$ for $i > 0$ so that

$$H^j \left(X, \mathcal{O}_X(m(K_X + S + F_A)) \right) = H^j \left(Z, \mathcal{O}_Z(m(K_Z + \mu_*(S + F_A))) \right).$$

By the Theorem on Formal Functions, it suffices to show that

$$H^i(X_n, L^{[m]}|_{X_n}) = 0$$

for all $i > 0$ and n , where X_n is the n^{th} formal neighborhood of X_1 in X . The fibration $X_1 \rightarrow C_1$ extends to a fibration $X_n \rightarrow C_n$ with all fibers isomorphic to E , where C_n is isomorphic to the n^{th} formal neighborhood of the component C_1 in C . That is, C_n is a rational curve with two embedded points of length n , and is locally isomorphic to $k[x, y]/(xy, y^n)$ around these points. Furthermore, $L|_{X_n} \cong \mathcal{O}_{X_n}(S_n)$, where S_n is a formal neighborhood of the section.

Lemma 7.4. *Let $f_n : X_n \rightarrow C_n$ be an elliptic fibration with all fibers isomorphic to E over a rational curve C_n with finitely many embedded points locally isomorphic to $k[x, y]/(xy, y^n)$. Let S_n be a section. Then $H^i(X_n, \mathcal{O}_{X_n}(mS_n)) = 0$ for any $m, n \geq 1$ and $i > 0$.*

Proof. A direct computation on E shows that

$$H^i(E, mP) = 0$$

for $i > 0$ and $m \geq 1$ where $P = (S_n)|_E$ is a point. Therefore $R^i(f_n)_*(\mathcal{O}_{X_n}(mS_n)) = 0$ for $i > 0$. Similarly,

$$h^0(E, mP) = m$$

so $R_{m,n} := (f_n)_*(\mathcal{O}_{X_n}(mS_n))$ is a rank m vector bundle.

For $m, n = 1$, the pushforward $(f_1)_*(\mathcal{O}_X(S))$ is a line bundle on $C_1 \cong \mathbb{P}^1$ with a section coming from pushing forward the section $\mathcal{O}_{X_1} \rightarrow \mathcal{O}_{X_1}(S_1)$. Therefore $H^i(C_1, R_{1,1}) = 0$ for $i > 0$. Pushing forward the exact sequence

$$0 \rightarrow \mathcal{O}_{X_1}((m-1)S_1) \rightarrow \mathcal{O}_{X_1}(mS_1) \rightarrow \mathcal{O}_{S_1}(mS_1|_{S_1}) \rightarrow 0$$

and noticing that $S_1|_{S_1} = 0$, we get

$$0 \rightarrow R_{m-1,1} \rightarrow R_{m,1} \rightarrow \mathcal{O}_{C_1} \rightarrow 0.$$

Since $H^i(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 0$ for $i > 0$, then $H^i(C_1, R_{m,1}) = 0$ for $i > 0$ by induction on m .

Now consider the ideal sequence

$$0 \rightarrow I_n \rightarrow \mathcal{O}_{C_n} \rightarrow \mathcal{O}_{C_{n-1}} \rightarrow 0$$

where I_n is torsion supported on finitely many points. Applying $(-) \otimes_{C_n} R_{m,n}$ and using base change for the Cartesian square

$$\begin{array}{ccc} E \times C_{n-1} & \xrightarrow{j} & E \times C_n \\ f_{n-1} \downarrow & & \downarrow f_n \\ C_{n-1} & \xrightarrow{i} & C_n \end{array}$$

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gives an exact sequence

$$0 \rightarrow K_{m,n} \rightarrow R_{m,n} \rightarrow R_{m,n-1} \rightarrow 0$$

where $K_{m,n}$ is torsion supported on finitely many points. Now by induction on n and the previous vanishing for $R_{m,1}$, we obtain $H^i(C_n, R_{m,n}) = 0$ for all $i > 0$. The required vanishing then follows from the Leray spectral sequence. \square

This shows it suffices to prove vanishing in Case 1 for the \mathcal{A} -broken elliptic surface pair $(Z, f_*(S + F_{\mathcal{A}}))$ after contracting the component X_1 . Applying this inductively, we can assume that in Case 1, the divisor $K_X + S + F_{\mathcal{A}}$ is big on every component.

Step 3: Next we reduce to the case when $K_X + S + F_{\mathcal{A}}$ has positive degree on every component of the section. Let $(X_0 \rightarrow C_0, S_0)$ be an elliptically fibered component such that $(K_X + S + F_{\mathcal{A}}).S_0 = 0$.

Let $\mu : X \rightarrow Z$ be the morphism contracting S_0 . Then $(Z, f_*(S + F_{\mathcal{A}}))$ is an \mathcal{A} -broken elliptic surface pair and

$$\mu_* \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}})) = \mathcal{O}_Z(m(K_Z + \mu_*(S + F_{\mathcal{A}})))$$

by Proposition 6.4. We want to show that

$$R^i \mu_* \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}})) = 0$$

for $i > 0$. This follows by Proposition 6.8, since the exceptional locus of μ is a rational curve $S_0 \cong \mathbb{P}^1$ with $S_0^2 < 0$ and $(K_X + S + F_{\mathcal{A}}).S_0 = 0$.

Step 4: We complete the proof in Case 1, under the assumption that $K_X + S + F_{\mathcal{A}}$ is big on every irreducible component of X , and has positive degree on every component of the section.

Proposition 7.5. *Let $(f : X \rightarrow C, S + F_{\mathcal{B}})$ be a \mathcal{B} -broken stable elliptic surface. Let $L^{[m]}$ denote the divisor $m(K_X + S + F_{\mathcal{A}})$ for $m \geq 2$, where $0 \leq \mathcal{A} \leq \mathcal{B}$. Suppose that $K_X + S + F_{\mathcal{A}}$ is nef, \mathbb{Q} -Cartier and big on every irreducible component of X and that $(K_X + S + F_{\mathcal{A}}).S_0 > 0$ for every component S_0 of S . Suppose that $F_{\mathcal{A}}|_Y = F_{\mathcal{B}}|_Y$ for every pseudoelliptic tree $Y \subset X$. Finally suppose either (a) $p_g(C) \neq 1$, or (b) \mathcal{A} is not identically zero. Then $H^i(X, L^{[m]}) = 0$ for all $i > 0$.*

Proof. We will apply Fujino's Theorem 6.10 to $L^{[m]}$. We have that

$$L^{[m]}(- (K_X + S + F_{\mathcal{A}})) = \mathcal{O}_X((m-1)(K_X + S + F_{\mathcal{A}}))$$

is big and nef on every irreducible component of X by assumption. Therefore, to apply the theorem, it suffices to prove that $K_X + S + F_{\mathcal{A}}$ is big on every slc center of $(X, S + F_{\mathcal{A}})$. This is clear for zero dimensional slc centers.

The one dimensional slc centers of $(X, S + F_{\mathcal{A}})$ are (a) the components of the section S , (b) the fibers F_j that are marked with coefficient $a_j = 1$, (c) and the components of the double locus D . Now $K_X + S + F_{\mathcal{A}}$ is big on every component of the section by assumption, and

$$(K_X + S + F_{\mathcal{A}}).F_j = 1/d$$

where F_i supports a possibly nonreduced fiber of multiplicity d . Here we have used the fact that a fiber with coefficient 1 is irreducible so that $F_{\mathcal{A}}.F_i = 0$.

This leaves case (c), the double locus D , which consists of three types of irreducible components:

- (i) For a stable or twisted (pseudo)fiber F along which an elliptic or type II pseudoelliptic is glued to the rest of X , we have

$$(K_X + S + F_{\mathcal{A}}).F = 1/d > 0;$$

- (ii) For every isotrivial component Z with $j = \infty$, there is the self intersection locus B . If Z is a pseudoelliptic component, then the morphism $Z' \rightarrow Z$ contracting the section of the associated elliptic component is an isomorphism in a neighborhood of B so we may suppose that Z is elliptic. In this case B is a section of Z and

$$(K_X + S + F_{\mathcal{A}}).B > 0.$$

- (iii) For every pseudoelliptic tree Y , there is the component M along which the root component Y_0 is attached to the rest of X . By the assumption

$$(K_X + S + F_{\mathcal{A}})|_Y = (K_X + S + F_{\mathcal{B}})|_Y$$

is ample on Y . In particular, $(K_X + S + F_{\mathcal{A}})|_Y.M > 0$.

Therefore $K_X + S + F_{\mathcal{A}}$ is big and nef on each slc stratum of $(X, F_{\mathcal{A}})$. Applying Theorem 6.10 we have the required vanishing

$$H^i\left(X, \mathcal{O}_X(m(K_X + F_{\mathcal{A}}))\right) = 0, \quad i > 0.$$

□

Step 5: Now we consider Case 2 of a pseudoelliptic tree Y indexed by a rooted tree $(T, 0)$. If $(Y, F_{\mathcal{A}})$ is already a stable pair, then we are done. Otherwise, there is some Y_{α} where the coefficients have been reduced. This implies the coefficients have been reduced on Y_{β} for any $\beta \leq \alpha$ as well.

Suppose Y_{α} is a leaf of the tree and that Y' is the union of Y_{β} for $\beta \neq \alpha$, i.e. the pseudoelliptic tree with dual graph $(T \setminus \alpha, 0)$. Suppose Y_{α} is attached to Y' along M a component of an intermediate fiber on Y' with reduced component A . Since the coefficients of Y_{α} have been reduced, then $L^{[m]}|_{Y'}(-M)$ and $L^{[m]}|_{Y_{\alpha}}$ are nef and \mathbb{Q} -Cartier by Lemma 7.3. By the attaching sequence, it suffices to show that

$$H^i(Y_{\alpha}, L^{[m]}|_{Y_{\alpha}}) = H^i(Y', L^{[m]}|_{Y'}(-M)) = 0.$$

By induction on the number of leaves of T , it suffices to prove that

$$H^i(Y, L^{[m]}|_Y(-M)) = 0$$

where $(Y, S + F_{\mathcal{A}})$ is a pseudoelliptic tree, M is a sum of the supports of finitely many arithmetic genus 1 components of intermediate pseudofibers of Y , and either

- (1) Y is irreducible, or
- (2) for each leaf $\alpha \in T$, $F_{\mathcal{A}}|_{Y_{\alpha}} = F_{\mathcal{B}}|_{Y_{\alpha}}$.

That is, we have separated all of the leaves on which coefficients have been decreased. Therefore, we have reduced to proving vanishing on the leaves themselves, as well as on a pseudoelliptic tree for which the coefficients of all emanating leaves have *not* been decreased.

Step 6: Let (Y, F_A) be a pseudoelliptic tree with dual graph $(T, 0)$ and suppose that $F_A|_{Y_\beta} = F_B|_{Y_\beta}$ for each leaf β , that is, we are in case (2) of Step 5 above. If $F_\beta|_Y = F_\alpha|_Y$ then $(K_X + S + F_\alpha)|_Y$ is ample so we're done. Thus suppose that there exists a component Y_α with $F_A|_{Y_\alpha} < F_B|_{Y_\alpha}$. We may take α to be maximal so that $F_A|_{Y_\beta} = F_B|_{Y_\beta}$ for all $\beta > \alpha$.

Let $\beta \in \alpha[1]$ (Definition 5.6) and $T_{\geq \beta} = \{\gamma \in V(T) : \gamma \geq \beta\}$ the subtree of T with root β (see Figure 8). Then $Y_{\geq \beta} = \bigcup_{\gamma \in T_{\geq \beta}} Y_\gamma$ is a pseudoelliptic subtree of Y with root component Y_β . Denoting by Y' the union of components of Y not in $Y_{\geq \beta}$, then $Y_{\geq \beta}$ is attached to Y' by gluing an irreducible pseudofiber M on $Y_{\geq \beta}$ to the arithmetic genus 1 component of an intermediate pseudofiber $M \cup A$ on $Y_\alpha \subset Y'$.

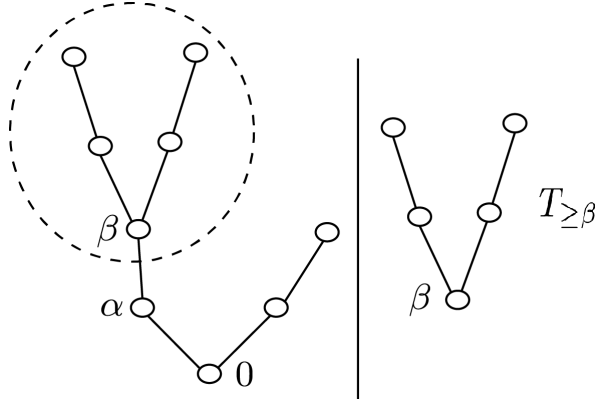


FIGURE 8. The rooted subtree $(T_{\geq \beta}, \beta)$ corresponds to the pseudoelliptic tree obtained by separating Y_β from Y_α .

We consider the gluing sequence

$$0 \rightarrow L^{[m]}|_{Y_{\geq \beta}}(-M) \rightarrow L^{[m]} \rightarrow L^{[m]}|_{Y'} \rightarrow 0.$$

Lemma 7.6. $L^{[m]}|_{Y_{\geq \beta}}(-M)$ is ample on $Y_{\geq \beta}$ and $L^{[m]}|_{Y'}$ is nef with positive degree on M .

Proof. Since no coefficients have been reduced on $Y_{\geq \beta}$, then $L|_{Y_{\geq \beta}}$ is ample so $L^{[m]}|_{Y_{\geq \beta}}(-M)$ is ample for m large enough. By assumption $\text{Coeff}(A, F_A) = \text{Coeff}(A, F_B)$ so that in particular $L \cdot M > 0$ since $(K_X + S + F_A)|_Y$ is ample. \square

Thus, we have that $H^i(Y_{\geq \beta}, L^{[m]}|_{Y_{\geq \beta}}(-M)) = 0$ for $i > 0$ so $H^i(Y, L^{[m]}) = H^i(Y', L^{[m]}|_{Y'})$. Therefore it suffices to prove vanishing for $L^{[m]}|_{Y'} = \mathcal{O}_Y(m(K_Y + M + F_A|_{Y'}))$, where $(Y', M + F_A|_{Y'})$ is a pseudoelliptic tree with a leaf Y_α such that coefficients on Y_α have been reduced. By induction on the number of leaves, we may suppose $(Y', F_A + M)$ is a pseudoelliptic tree that $F_A|_{Y_\alpha} < F_B|_{Y_\alpha}$ for every leaf α and M is a sum of reduced arithmetic genus 1 components of intermediate pseudofibers on the leaf components. Furthermore, by the above lemma, $(K_Y + F_A + M) \cdot M_0 > 0$ for each component M_0 of M .

Since $F_{\mathcal{A}}|_{Y_\alpha} < F_{\mathcal{B}}|_{Y_\alpha}$ for every leaf α , we can apply Step 5 to the pseudoelliptic tree $(Y, F_{\mathcal{A}} + M)$. That is, we can separate the irreducible components of Y . This reduces to proving $H^i(Y, L^{[m]}|_Y(-M')) = 0$ for $i > 0$ where Y is an irreducible pseudoelliptic surface, M' is a union of the supports of arithmetic genus 1 components of intermediate fibers, and we can write

$$L^{[m]}|_Y(-M') = \mathcal{O}_Y \left(m \left(K_Y + F_{\mathcal{A}}|_Y + G + M + \frac{m-1}{m} M' \right) \right),$$

where M is a union of components of intermediate fibers with $L.M > 0$ and G is an irreducible fiber. Denoting $\Delta = F_{\mathcal{A}}|_Y + G + M + \frac{m-1}{m} M'$, we are then left to consider an irreducible pseudoelliptic pair (Y, Δ) .

Step 7: Let (Y, Δ) be an irreducible pseudoelliptic pair as in Step 5 case (1) or the conclusion of Step 6 above and suppose $K_Y + \Delta$ is big. Now we may take the partial log semi-resolution $\mu : X \rightarrow Y$ by the associated elliptic surface $(X \rightarrow C, S)$ over a necessarily rational curve.

We may write

$$K_X + S + F = \mu^*(K_Y + \Delta) + tS$$

for $0 \leq t \leq 1$ where $F = \mu_*^{-1} \Delta$ is a union of (not necessarily reduced) fiber components. By Proposition 6.4 we have

$$\mu_* \mathcal{O}_X(m(K_X + S + F)) = \mathcal{O}_Y(m(K_Y + \Delta)).$$

Proceeding as in Proposition 7.5, we aim to apply Fujino's Theorem 6.10. That is, we need to check that $K_Y + \Delta$ is big on each of the slc strata of (Y, Δ) . The divisor $K_Y + \Delta$ is big on Y by assumption, and it is trivially big on the zero dimensional log canonical centers. This leaves the one dimensional log canonical centers of (Y, Δ) . These are exactly the images of the log canonical centers of $(X, S + F_{\mathcal{A}})$, noting that the image of S is a point so we need not consider it. Now $(K_Y + \Delta).M > 0$ for any log canonical center supported on an intermediate pseudofiber by Lemma 7.6. Using the projection formula, it follows that $K_Y + \Delta$ is big on each of the other log canonical center by the computations in the proof of Proposition 7.5. Therefore

$$H^i(Y, \mathcal{O}_Y(m(K_Y + \Delta))) = 0,$$

for all $i > 0$ by Theorem 6.10.

In particular, this finishes the proof for the following cases, where we know that $K_Y + \Delta$ is big:

- Y is an irreducible pseudoelliptic with $\deg \mathbb{L} \geq 3$ (Proposition 5.14),
- Y is an irreducible pseudoelliptic with $\deg \mathbb{L} = 2$ with $\Delta \not\sim_{\mathbb{Q}} 0$ (Proposition 5.18).

We are left to deal with irreducible pseudoelliptics (Y, Δ) with $K_Y + \Delta$ *not* big.

Step 8: Let (Y, Δ) be a pseudoelliptic with $\deg \mathbb{L} = 1$ and Iitaka dimension $\kappa(K_Y + \Delta) = 1$. By Proposition 5.15, $K_Y + \Delta \sim_{\mathbb{Q}} \mu_* \Sigma$, where $\mu : Z \rightarrow Y$ is the contraction of the section of

the associated elliptic surface $f : Z \rightarrow C$ and Σ is a rational multisection of f disjoint from S . Since Σ is in the locus where μ is an isomorphism, it suffices to prove that $H^i(Z, \mathcal{O}_Z(m\Sigma)) = 0$.

By Lemma 7.7 below, $f_*\mathcal{O}_Z(m\Sigma)$ is a semipositive vector bundle on \mathbb{P}^1 . In particular,

$$H^i(\mathbb{P}^1, f_*\mathcal{O}_Z(m\Sigma)) = 0.$$

Furthermore, $G \cdot \Sigma > 0$ for any irreducible fiber G , since Σ is an effective multisection. In particular, $m\Sigma - (K_Z + S + F_A)$ is f -nef and f -big over each slc stratum of Y for $m \gg 1$. Therefore $R^i f_*\mathcal{O}_Z(m\Sigma) = 0$ by Fujino's Theorem 6.10 for $i > 0$ and $m \gg 0$, and so $H^i(Z, \mathcal{O}_Z(m\Sigma)) = 0$ by the Leray spectral sequence.

Lemma 7.7. *Let $f : Y \rightarrow C$ be a fibration from an irreducible slc surface to a reduced curve and let Σ be a multisection with $|\Sigma|$ basepoint free. Then $f_*\mathcal{O}_Y(m\Sigma)$ is a semipositive vector bundle on C for $m \gg 0$.*

Proof. Note that for a finite morphism $\varphi : B \rightarrow C$ and a vector bundle \mathcal{E} on C , the vector bundle \mathcal{E} is semipositive on C if and only if $\varphi^*\mathcal{E}$ is semipositive on B .

Since $R^i f_*\mathcal{O}_Y(m\Sigma) = 0$ for $i > 0$ and $m \gg 0$, we may apply cohomology and base change to conclude that $f_*\mathcal{O}_Y(m\Sigma)$ is a vector bundle and its formation commutes with basechange. Let $\nu : \tilde{C} \rightarrow C$ be the normalization. Consider the base change

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\mu} & Y \\ g \downarrow & & \downarrow f \\ \tilde{C} & \xrightarrow{\nu} & C \end{array}$$

Since ν is finite, it suffices to prove that $\nu^* f_*\mathcal{O}_Y(m\Sigma) \cong g_*\mathcal{O}_Y(m\mu^*\Sigma)$ is semipositive. Since $\mu^*\Sigma$ is a multisection of g , we may assume without loss of generality that C is smooth.

Since $|m\Sigma|$ is basepoint free, there exists a general member D that is a reduced divisor, i.e. $D = \sum_{i=1}^s C_i$ where each C_i is a distinct multisection. There exists a finite base change

$$\begin{array}{ccc} Y' & \xrightarrow{\phi} & Y \\ g \downarrow & & \downarrow f \\ C' & \xrightarrow{\varphi} & C \end{array}$$

such that $\phi^*C_i \sim_{\mathbb{Q}} \sum_j S_{ij}$ is a sum of distinct sections. Therefore we may assume that $m\Sigma \sim_{\mathbb{Q}} D = \sum_{i=1}^s C_i$ is a finite sum of distinct sections.

Now we use induction on the number of sections. Let $D_k = \sum_{i=1}^k C_i$. Then for $k = 1$, D_1 is a single section and $f_*\mathcal{O}_Y(D_1)$ is a line bundle with a section induced by pushing forward the section $\mathcal{O}_Y \rightarrow \mathcal{O}_Y(D_1)$. Therefore $f_*\mathcal{O}_Y(D_1)$ is semipositive.

Now let $k \geq 2$. We consider the exact sequence

$$0 \rightarrow \mathcal{O}_Y(D_{k-1}) \rightarrow \mathcal{O}_Y(D_k) \rightarrow \mathcal{O}_{C_k}(D_{k-1}|_{C_k}) \rightarrow 0$$

induced by adding C_k . Since the sections are all distinct, then $D_{k-1} \cdot C_k \geq 0$ so that $\mathcal{O}_{C_k}(D_{k-1}|_{C_k})$ is a semipositive line bundle. Pushing forward, and noting that

$R^1 f_* \mathcal{O}_Y(D_{k-1}) = 0$, then the sequence

$$0 \rightarrow f_* \mathcal{O}_Y(D_{k-1}) \rightarrow f_* \mathcal{O}_Y(D_k) \rightarrow f_* \mathcal{O}_{C_k}(D_{k-1}|_{C_k}) \rightarrow 0$$

is exact. The first term is semipositive by the inductive hypothesis, and the last term is semipositive since f is an isomorphism on C_k . Therefore the middle term is semipositive. \square

Step 9: Finally, we are left with the case of an irreducible pseudoelliptic (X, Δ) with Iitaka dimension $\kappa(K_X + \Delta) = 0$, which occurs for $\deg \mathbb{L} = 1$ and $\deg \mathbb{L} = 2$ as in Propositions 5.15 and 5.18. We have $K_X + \Delta \sim_{\mathbb{Q}} 0$ so for m large and divisible enough, the sheaf $\mathcal{O}_X(m(K_X + \Delta)) = \mathcal{O}_X$. If X is normal, then it is either a rational or K3 surface and $H^i(X, \mathcal{O}_X) = 0$ for $i > 0$.

If X is not normal, then the associated elliptic surface $f : Y \rightarrow \mathbb{P}^1$ is an isotrivial elliptic fibration with $j = \infty$. Consider a one parameter family

$$\begin{array}{ccc} \mathcal{X} & \xrightarrow{g} & \mathcal{C} \\ & \searrow \pi & \swarrow \\ & T & \end{array}$$

of elliptic surfaces over a smooth curve T such that $g_0 : \mathcal{X}_0 \rightarrow \mathcal{C}_0$ is isomorphic to $f : Y \rightarrow \mathbb{P}^1$, and $g_t : \mathcal{X}_t \rightarrow \mathcal{C}_t$ is an isotrivial elliptic fibration with $\deg \mathbb{L} = 1$ and $j \neq \infty$ for $t \neq 0$. Then

$$H^2(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$$

for all $t \in T$ by Proposition 6.11 and $\dim H^0(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 1$ for all $t \in T$ since \mathcal{X}_t is an integral projective variety. Therefore $R^1 \pi_* \mathcal{O}_{\mathcal{X}}$ is a locally free sheaf that satisfies base change and

$$(R^1 \pi_* \mathcal{O}_{\mathcal{X}})_t = H^1(\mathcal{X}_t, \mathcal{O}_{\mathcal{X}_t}) = 0$$

for $t \neq 0$ by above. It follows that

$$(R^1 \pi_* \mathcal{O}_{\mathcal{X}})_0 = H^1(Y, \mathcal{O}_Y) = 0.$$

This concludes the proof of Theorem 7.1. \square

Theorem 7.8. (*Invariance of log plurigenera*) Let $\pi : (X \rightarrow C, S + F_{\mathcal{B}}) \rightarrow B$ be a family of \mathcal{B} -stable broken elliptic surfaces over a reduced base B . Let $0 \leq \mathcal{A} \leq \mathcal{B}$ such that $K_{X/B} + S + F_{\mathcal{A}}$ is a π -nef and \mathbb{Q} -Cartier divisor. Then $\pi_* \mathcal{O}_X(m(K_{X/B} + S + F_{\mathcal{A}}))$ is a vector bundle on B whose formation is compatible with base change $B' \rightarrow B$ for $m \geq 2$ divisible enough.

Proof. If either (a) $p_g(C_b) \neq 1$, or (b) \mathcal{A} is not identically zero, then we may apply Theorem 7.5 to see that $H^i(X_b, m(K_{X/B} + S + F_{\mathcal{A}})|_{X_b}) = 0$ for $i > 0$ and for all closed points $b \in B$ so the result follows by the proper base change theorem.

Suppose $p_g(C_b) = 1$ and $\mathcal{A} = (0, \dots, 0)$ is identically zero. We may suppose that $\mathcal{B} = (a, 0, \dots, 0)$ has exactly one nonzero entry by applying the above result and first decreasing all but one coefficient to 0. Then (C_b, ap) is a one pointed stable genus 1 curve with $a < 1$. In particular, it is irreducible. Therefore X_b contains a single elliptically fibered

component $X_0 \rightarrow C_b$ with a marked divisor $(F_a)_b$ lying over p . There are three cases to consider:

- (i) X_0 is properly elliptic and $F_a = aF$ is a reduced marked fiber,
- (ii) X_0 is properly elliptic and there is a pseudoelliptic tree $(Y_b, (F_a)_b|_{Y_b})$ attached to an intermediate fiber $E \cup A$ above p and $(F_a)_b|_{X_0} = aA$, or
- (iii) $\deg \mathbb{L} = 0$ and $X_b = X_0 = C_b \times E_b$ is a product.

In either of case (i) and (ii) there may be unmarked type I or II pseudoelliptics attached elsewhere to X_0 .

Let us denote $L^{[m]} := \mathcal{O}_X(m(K_{X/B} + S))$. The linear series $|L_b^{[m]}|$ is semi-ample by Proposition 6.2 and $L_b^{[m]}.S_b = 0$. Thus the linear series factors through the contraction of S_b which gives a morphism $\mu : X_b \rightarrow Z_b$. In case (i) and (ii) this maps onto an slc broken pseudoelliptic surface with an elliptic singularity at $\mu(S_b)$ and $\mu_*L_b^{[m]} = \mathcal{O}_{Z_b}(m(K_{Z_b}))$.

In case (i), the pair (X_0, S_b) is log general type by Proposition 5.14 and for every other component $W \subset X_b$, the line bundle $L_b^{[m]}|_W = \mathcal{O}_{X_b}(m(K_{X_b} + S_b + (F_a)_b))|_W$ is still ample on W . It follows that K_{Z_b} is big and nef on each slc stratum of $(Z_b, 0)$ so $H^1(Z_b, \mu_*L_b^{[m]}) = 0$ by Theorem 6.10.

In case (ii) we consider the attaching sequence

$$0 \rightarrow \mu_*L_b^{[m]}|_{Z'_b}(-M) \rightarrow \mu_*L_b^{[m]} \rightarrow L_b^{[m]}|_{Y_b} \rightarrow 0$$

where $M = \mu_*E$ is the curve along which Y_b is attached to $Z_0 = \mu(X_0)$ and Z'_b is the union of components of Z_b not contained in Y_b . Now $L_b|_{Y_b} = K_{Y_b} + E$ and (Y_b, E) is a broken pseudoelliptic tree and we can apply Theorem 7.1 to conclude $H^1(Y_b, L_b^{[m]}|_{Y_b}) = 0$. On the other hand, $K_{Z_b}|_{Z'_b} = K_{Z'_b} + M$ so

$$\mu_*L_b^{[m]}|_{Z'_b}(-M) = \mathcal{O}_{Z'_b} \left(m(K_{Z'_b} + \frac{m-1}{m}M) \right).$$

As in case (i), the divisor $K_{Z'_b} + \frac{m-1}{m}M$ is big and nef on every slc stratum of $(Z'_b, \frac{m-1}{m}M)$ so

$$H^1(Z'_b, \mu_*L_b^{[m]}|_{Z'_b}(-M)) = 0$$

by Theorem 6.10 and we conclude that $H^1(Z_b, \mu_*L_b^{[m]}) = 0$.

In either case (i) or (ii), it follows that $H^1(X_b, L_b^{[m]}) = H^0(Z_b, R^1\mu_*L_b^{[m]})$. Now

$$(K_{X_b} + S_b)|_{S_b} \sim_{\mathbb{Q}} 0$$

so $L_b^{[m]}|_{S_b} = \mathcal{O}_{S_b}$ for m divisible enough. On the other hand S_b is an irreducible nodal arithmetic genus 1 curve so by the theorem on formal functions, $R^1\mu_*L_b^{[m]}$ is a skyscraper sheaf supported on $\mu(S_b)$ with 1 dimensional fiber and $h^1(X_b, L_b^{[m]}) = 1$.

In case (iii), consider the trivial fibration $f : X_b \rightarrow C_b$ with section S_b and $g(C_b) = 1$. Then $K_{X_b} = 0$ and $L_b^{[m]} = \mathcal{O}_{X_b}(mS_b)$. Furthermore, $R^1f_*\mathcal{O}_{X_b}(mS_b) = 0$ and $f_*\mathcal{O}_{X_b}(mS_b) = \mathcal{O}_{C_b}^{\oplus m}$

for $m \geq 1$ by [Mir80, II.3.5 and II.4.3]. It follows that $h^1(X_b, L_b^{[m]}) = h^1(C_b, \mathcal{O}_{C_b}^{\oplus m}) = m$.

In each case $h^1(X_b, \mathcal{O}_X(m(K_{X/B} + S))|_{X_b})$ is constant and $h^2(X_b, \mathcal{O}_X(m(K_{X/B} + S))|_{X_b}) = 0$ by Proposition 6.11. Therefore $h^0(X_b, \mathcal{O}_X(m(K_{X/B} + S))|_{X_b})$ is constant by invariance of $\chi(X_b, \mathcal{O}_X(m(K_{X/B} + S))|_{X_b})$. Thus we can apply cohomology and base change over a reduced scheme to conclude that $\pi_* \mathcal{O}_X(m(K_{X/B} + S))$ is a vector bundle whose formation is compatible with base change. \square

Remark 7.9. Note that for the first part of Theorem 7.8, we do *not* need to assume that B is reduced. Indeed, whenever we can apply the vanishing theorem 7.5, a strong form of proper base change ensures that the formation of $\pi_* \mathcal{O}_X(m(K_{X/B} + S + F_{\mathcal{A}}))$ commutes with arbitrary base change for any base B . It is only in the second case when the higher cohomology *does not* vanish that we need to assume B is reduced to apply cohomology and base change. This will not matter in the sequel as we restrict to normal base schemes.

The above Theorem 7.8 allows us to compute the \mathcal{A} -stable model of a \mathcal{B} -stable family by working fiber by fiber. This is used in the next section to explicitly describe the steps of the log MMP to compute the stable limits of a 1-parameter family. Then in section 9, we use Theorem 7.8 to show that performing the steps of the log MMP on a family of elliptic surfaces is functorial. This leads to the existence of reduction morphisms between moduli spaces of elliptic surfaces for weights $0 \leq \mathcal{A} \leq \mathcal{B}$ as above.

8. STABLE REDUCTION

The goal of this section is to prove a stable reduction theorem for \mathcal{A} -broken elliptic surfaces in the spirit of La Nave [LN]. As a result we obtain properness of the moduli spaces $\mathcal{E}_{v, \mathcal{A}}$ and give a description of the surfaces that appear in the boundary.

Our strategy for stable reduction is to first compute stable limits of a family of irreducible elliptic surfaces with large coefficients. To this end, in [AB16b] we use the theory of *twisted stable maps* to compute stable limits in the case when all singular fibers are marked with coefficient $b_i = 1$. We then run the minimal model program while reducing the coefficients to compute the stable limit for weights \mathcal{A} using the classification of log canonical models of elliptic surfaces in [AB16a] as well Theorem 7.8.

8.1. Wall and chamber structure. Let $\mathcal{D} \subset (\mathbb{Q} \cap [0, 1])^n$ be the set of *admissible weights*: weight vectors \mathcal{A} such that $K_X + S + F_{\mathcal{A}}$ is pseudoeffective. A *wall and chamber* decomposition of \mathcal{D} is a finite collection \mathcal{W} of hypersurfaces (the *walls*), and the *chambers* are the connected components of the complement of \mathcal{W} in \mathcal{D} .

First we describe a wall and chamber decomposition of \mathcal{D} defined by where the log canonical model of an \mathcal{A} -slc elliptic surface changes as \mathcal{A} varies. The collection of walls \mathcal{W} corresponds to the steps in the MMP required to produce a stable limit of a family of elliptic surfaces over a smooth curve.

Definition 8.1. The collection \mathcal{W} consists of the following types of walls:

- I A wall of *Type* W_I is a wall arising from the log canonical transformations seen in Section 5– that is, the walls where the fibers of the relative log canonical model transition from twisted, to intermediate, to Weierstrass fibers.
- II A wall of *Type* W_{II} is a wall at which the morphism induced by the log canonical contracts the section of some components. By Corollary 5.12 these are the same as the walls for Hassett space $\overline{\mathcal{M}}_{g,\mathcal{A}}$.
- III A wall of *Type* W_{III} is a wall where the morphism induced by the log canonical contracts a rational pseudoelliptic component. These are determined by Proposition 5.15 and Remark 5.16

There are also *boundary walls* given by $a_i = 0, 1$ at the boundary of \mathcal{D} . These can be of any of the types above.

Proposition 8.2. *The non-boundary walls of each type are described as follows:*

(a) *Type W_I walls are defined by the equations*

$$a_i = \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}.$$

(a) *Type W_{II} walls are defined by equations*

$$\sum_{j=1}^k a_{i_j} = 1.$$

where $\{i_1, \dots, i_k\} \subset \{1, \dots, n\}$. When the base curve is rational there is another W_{II} wall at

$$\sum_{i=1}^r a_i = 2.$$

(a) *Type W_{III} walls are determined by finitely many equations depending on the numerical invariants and types of singular fibers that appear on the moduli space.*

In particular, there are only finitely many walls and chambers.

Proof. Part (a) follows from the results of Section 5 (see [AB16a] for more details) since these are exactly the coefficients at which type N_1, II, III and IV fibers transition from Weierstrass models to intermediate models.

Part (b) follows from Proposition 5.11 since $(K_X + S + F_{\mathcal{A}}).S > 0$ if and only if the base curve is a weighted stable pointed curve. When $\sum a_{i_j} = 1$, the section of any component fibered over a rational curve, which is attached to the other components of the surface along one attaching fiber, and contains marked fibers i_1, \dots, i_k gets contracted. When the base curve is \mathbb{P}^1 and $\sum a_i = 2$, the section of *every* elliptic surface gets contracted so that all \mathcal{A} -slc elliptic surfaces have *only* pseudoelliptic components.

For Part (c), note that Type W_{III} walls occur whenever $K_X + S + F_{\mathcal{A}}$ is *not big* on a rational (pseudo)elliptic component. Let Y be a minimal log resolution of this component so that Y is a smooth rational elliptic surface with section S_Y , marked fibers F_{I_1}, \dots, F_{I_k} , and attaching fiber G . Since the volume v is fixed and the family of elliptic surfaces with fixed

volume is bounded [Ale94, Theorem 9.2], there are only finitely many combinations of fiber types for $F_{I_1}, \dots, F_{i_k}, G$. The Type W_{III} walls then occur when

$$\left(K_Y + S_Y + \sum_j a_{i_j} F_{i_j} + G \right)^2 = 0$$

by proposition 6.7. This gives finitely many equations. \square

Question 1. *Are the type W_{III} walls given by linear equations?*

The equations defining the W_{III} appear quadratic in a_i . However in all computations we have carried out, these quadratic equations factor yielding linear walls.

8.2. The birational transformations across each wall. We wish to describe the birational transformations that a family of \mathcal{A} -broken stable elliptic surfaces undergoes as \mathcal{A} crosses a wall. Let $(f : X \rightarrow C, S + F_{\mathcal{A}}) \rightarrow B$ be a one parameter family of broken elliptic surfaces with normal generic fiber and special fiber $f' : X' \rightarrow C'$.

8.2.1. Type W_{I} . If F is the support of a non-reduced fiber, then at the wall at coefficient $a_0 = 0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}$ or $\frac{5}{6}$ (depending on the Kodaira fiber type), X undergoes a divisorial contraction where F transforms from intermediate to Weierstrass. Similarly, at the boundary wall $a_1 = 1$, the surface X undergoes a divisorial contraction where F transforms from an intermediate into a twisted fiber.

8.2.2. Type W_{II} . Let \mathcal{A}_0 be a weight on the non-boundary wall defined by $\sum a_{i_j} = 1$ for $\{i_1, \dots, i_k\}$. Let \mathcal{A}_{\pm} be in the adjacent chambers with $\sum a_{i_j} = 1 \pm \epsilon$ for ϵ very small. \mathcal{A}_0 is on a wall for the Hassett space where a leaf component C'_0 of the central fiber C' is contracted.

La Nave studied this situation in [LN, Section 4.3]. At A_0 , the section S'_0 of an elliptic component X'_0 lying over C' in the central fiber $X' \rightarrow C'$ of the \mathcal{A}_+ stable family $X \rightarrow C$ must contract by Proposition 5.11. This is a log canonical contraction of the pair $(X, S + F_{\mathcal{A}_0})$, but it is an extremal contraction of the pair $(X, S + F_{\mathcal{A}_-})$.

Since the total space X is a threefold and S'_0 is a curve, this is a small contraction so we must perform a flip to compute the \mathcal{A}_- stable model. La Nave computes this flip explicitly using a local toric model around S'_0 inside the total space X [LN, Theorem 7.1.2]. This leads to the formation of a type I pseudoelliptic surface Z in the central fiber attached to the component E of an intermediate (pseudo)fiber $E \cup A$ where A is the flipped curve, as depicted in Figure 9:

At a boundary type W_{II} wall, a rational component C'_0 of C' which is not a leaf may contract. The contraction of the corresponding section component S'_0 in the central fiber X' of $X \rightarrow B$ is a log canonical contraction which forms a type II pseudoelliptic surface.

Finally when the genus of the base curve is 0, we must consider the wall defined by $\sum a_i = 2$. In this case the base curve is contracted to a point and so the section of the total family $X \rightarrow C \rightarrow B$ is contracted by a divisorial log canonical contraction. This produces a one parameter family of pseudoelliptic surfaces $Z \rightarrow B$ with normal generic fiber and special fiber consisting of an \mathcal{A} -broken pseudoelliptic surface.

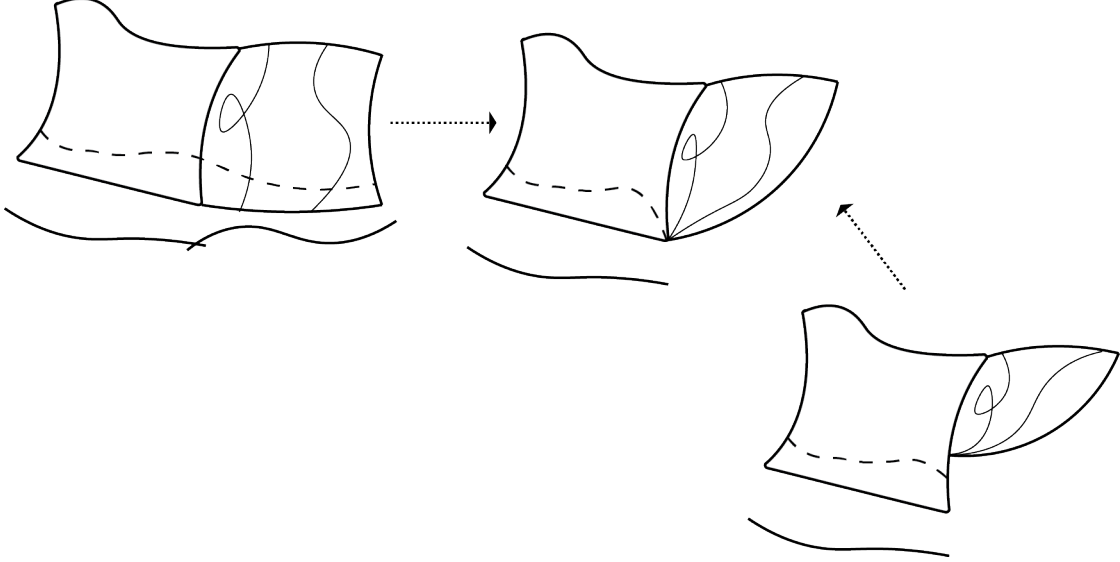


FIGURE 9. This depicts, from left to right, the central fiber of the \mathcal{A}_+ , \mathcal{A}_0 and \mathcal{A}_- stable families where \mathcal{A}_0 is a type W_{II} wall.

8.2.3. *Type W_{III} .* At \mathcal{A}_0 , there is a pseudoelliptic component Z in the central fiber of X' for which $K_X + S + F_{\mathcal{A}_0}$ is nef but not big. Then the total space $(X, S + F_{\mathcal{A}_0})$ undergoes a divisorial log canonical contraction $X \rightarrow Y$ which contracts Z onto either a point or a curve as determined by Remark 5.16.

When Z contracts to a point, this results in a cuspidal cubic fiber on the central fiber $Y' \rightarrow C'$ of Y at the point that Z contracted to. When \mathcal{A}_0 is not on a boundary wall, then the surface Y' has at worst a rational singularity at this point by Proposition 5.17. At a boundary, the contraction of Z may produce an elliptic singularity at the cusp.

8.2.4. *Multiple walls.* Figures 10, 11 and 12 illustrate some of the multi-step transformations the central fiber can undergo due to the birational transformations of X when crossing several walls at once.

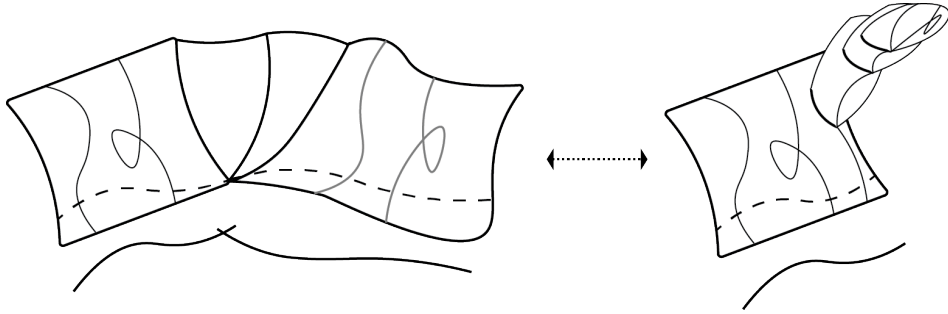


FIGURE 10. Here a type W_{II} wall is crossed which causes the right most component to transform into a type I pseudoelliptic. However, that then causes the type II pseudoelliptics to also become type I since they have no marked fibers.

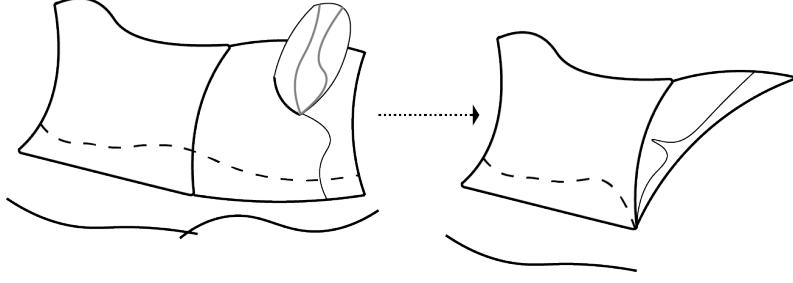


FIGURE 11. This is a simultaneous W_{II} and W_{III} wall where the type I pseudoelliptic component contracts onto a point and the right most elliptic component becomes a pseudoelliptic.

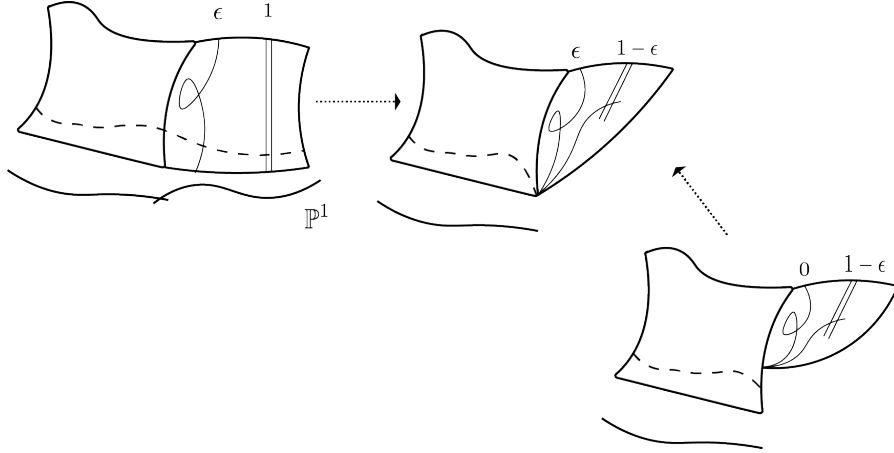


FIGURE 12. This is a simultaneous W_I and W_{II} where the twisted fiber becomes an intermediate fiber and a type I pseudoelliptic forms.

8.3. Explicit stable reduction. Recall the following definition (see Definition 4.9 [AB16b]):

Definition 8.3. An \mathcal{A} -broken elliptic surface $(f : X \rightarrow C, S + F_{\mathcal{A}})$ is **twisted** if $a_i = 1$ for all i , there are no pseudoelliptic components, and the support of every non-reduced fiber is contained in $\text{Supp}(F_{\mathcal{A}})$.

In [AB16b], we used the Abramovich-Vistoli moduli space of twisted stable maps [AV02] to construct a proper moduli space of twisted elliptic surfaces analogous to the moduli spaces of fibered surfaces considered in [AV97]. This is the starting point for computing the stable limits in $\mathcal{E}_{v, \mathcal{A}}$ for any \mathcal{A} .

Given a family of \mathcal{A} -stable irreducible elliptic surface $(X \rightarrow C, S + F_{\mathcal{A}}) \rightarrow U$ over a punctured curve U , the idea is to

- (1) increase the coefficients so that $a_i = 1$ for all i , and
- (2) add the supports of any unstable fibers to the boundary divisor.

Then the stable model of this new pair will be a family of twisted elliptic surfaces. By the results of [AB16b], this family extends uniquely after a base change $U' \rightarrow U$. Finally, we can run the log MMP to compute the stable model as we decrease coefficients again.

This is analogous to the approach used by La Nave [LN] to compute stable limits of stable Weierstrass fibrations, i.e. when $\mathcal{A} = 0$.

Theorem 8.4. *The moduli stack $\mathcal{E}_{v,\mathcal{A}}$ is proper.*

Proof. Consider a family of normal \mathcal{A} -stable elliptic surfaces $(X^0, S^0 + F_{\mathcal{A}}^0) \rightarrow C^0 \rightarrow U$ over $U = B \setminus p$, a smooth curve minus one point. Let $\mathcal{B}_1 = (1, \dots, 1)$ be the constant weight 1 vector and let $G^0 = G_{r+1}^0 + \dots + G_s^0$ be the reduced divisor whose support consists of the singular fibers not contained in $\text{Supp}(F_{\mathcal{A}})$. Define $D_{\mathcal{B}}^0 = F_{\mathcal{B}_1}^0 + G^0$ so that $(X^0, S^0 + D_{\mathcal{B}}^0) \rightarrow C^0 \rightarrow U$ is a family of pairs with all non-stable fibers marked and all fibers marked with coefficient one. We index the weight vector $\mathcal{B} = (b_1, \dots, b_r, b_{r+1}, \dots, b_s)$ such that b_i for $i = 1, \dots, r$ are the coefficients of the original marked fibers F_i .

After performing a log resolution, we can take the log canonical model of this pair to obtain a family of slc elliptic surfaces $(X^1, S^1 + D_{\mathcal{B}}^1) \rightarrow C^0 \rightarrow U$, such that all fibers are either stable or twisted, and all fibers that are not of type I_n are contained in either the double locus of X or in $D_{\mathcal{B}}^1$. By [AB16b, Corollary 5.10], there is a map $C^0 \rightarrow \overline{M}_{1,1}$ making $(X^1, S^1 + D_{\mathcal{B}}^1) \rightarrow \overline{M}_{1,1}$ an Alexeev stable map from a twisted elliptic surface (see Section 5 and Proposition 5.2 of [AB16b]).

By [AB16b, Proposition 5.2], the moduli space of Alexeev stable maps from a twisted elliptic surface is proper. Therefore, after a finite base change $B' \rightarrow B$, this family extends uniquely to a family $(Z_1, S_1 + D_{\mathcal{B}}) \rightarrow C_1 \rightarrow B'$ of twisted elliptic surfaces over B' with a well defined j -invariant map $C_1 \rightarrow \overline{M}_{1,1}$. Furthermore the central fiber consists of only elliptic components fibered over a possibly reducible nodal curve.

Now consider the line segment $\mathcal{A}(t) := t\mathcal{B} + (1-t)\mathcal{A}_0$ for $t \in [0, 1]$ where $\mathcal{A}_0 = (a_1, \dots, a_r, 0, \dots, 0)$. By Proposition 8.2, there are finitely many $t_0 = 0, t_1, \dots, t_{n-1}, t_n = 1$ so that $\mathcal{A}(t_k)$ are on walls.

By invariance of log plurigena (Theorem 7.8), we can compute the stable model of $\pi : (Z_1, S_1 + D_{\mathcal{B}(t)}) \rightarrow C_1 \rightarrow B'$ as we decrease t from $t = 1$ by taking the stable model of each fiber as long as $K_{\pi} + S_1 + D_{\mathcal{B}(t)}$ remains π -nef, and \mathbb{Q} -Cartier. First we need that each wall-crossing preserves the structure of a fibered surface:

Lemma 8.5. *Let $(f : X \rightarrow C, S + F_{\mathcal{A}})$ be a \mathcal{B} -broken stable elliptic surface. Let $\mathcal{A} \leq \mathcal{B}$ and denote by X' and C' the stabilizations of X and C with respect to \mathcal{A} respectively. Then there exists a commutative diagram:*

$$\begin{array}{ccc} X & \longrightarrow & C \\ \downarrow & & \downarrow \\ X' & \longrightarrow & C' \end{array}.$$

Proof. Let $\phi : X \dashrightarrow X'$ be the log canonical birational map induced by $m(K_X + S + F_{\mathcal{A}})$. We can factor ϕ into a sequence of type W_I , W_{II} and W_{III} birational transformations described in Section 8. We reduce to checking that for each of these birational transformations, there is a compatible factorization of $X' \rightarrow C'$.

- I. If ϕ is a W_I type transformation, that is, a transition between twisted, intermediate and Weierstrass fibers, then ϕ is a composition of blowups and blowdowns of fiber components so there is a factorization $X' \rightarrow C$.
- II. If ϕ is a W_{II} type transformation, then there is a diagram

$$\begin{array}{ccc} X_- & & X_+ \\ & \searrow & \swarrow \\ & X_0 & \end{array}$$

where $X_+ \rightarrow X_0$ is the contraction of a section component $X_- \rightarrow X_0$ is birational on every component and ϕ is either $X_+ \rightarrow X_0$ or $X_+ \dashrightarrow X_-$ (see Section 10 for details). By Proposition 5.11, the map $X_+ \rightarrow X_0$ contracts a section component if and only if that component of the base curve is contracted by the morphism $C \rightarrow C'$. Therefore there is a unique factorization $X_0 \rightarrow C'$ also inducing a unique map $X_- \rightarrow C'$ by composition.

- III. If ϕ is a type W_{III} transformation, then it contracts components of X which are contracted to a point by $f : X \rightarrow C$. Therefore there is a unique factorization $X' \rightarrow C$.

□

Now for $0 < \epsilon \ll 1$, there exists a unique family of $\mathcal{A}(1 - \epsilon)$ -weighted stable elliptic surfaces $(Z_{1-\epsilon}, S_{1-\epsilon} + D_{\mathcal{A}(1-\epsilon)}) \rightarrow C_1 \rightarrow B'$ obtained by the blowup from twisted to intermediate models of the marked fibers. Then one performs the following whenever $\mathcal{A}(t)$ crosses a wall:

- Each time t crosses a type W_I or W_{III} wall t_k , the family undergoes a divisorial contraction as described in Sections 8.2.1 and 8.2.3. In this case one obtains a $\mathcal{A}(t_k)$ -weighted stable family $(Z_{t_k}, S_{t_k} + D_{\mathcal{A}(t_k)}) \rightarrow C_{t_k} \rightarrow B'$;
- Across a type W_{II} wall t_l , the map $X_t \rightarrow X_{t_l}$ is a flipping contraction of a section of a component of the central fiber. As described in Section 8.2.2 there is a unique flip $X_{t'} \rightarrow X_{t_l}$ constructed by La Nave in [LN] giving an $\mathcal{A}(t')$ -weighted stable family

$$(Z_{t'}, S_{t'} + D_{\mathcal{A}(t')}) \rightarrow C_{t'} \rightarrow B'.$$

Since there are only finitely many walls crossed, we eventually obtain an $\mathcal{A}(0) = \mathcal{A}_0$ -weighted stable family $\pi : (Z_\delta, S_0 + D_{\mathcal{A}_0}) \rightarrow C_0 \rightarrow B'$. Forgetting about the auxiliary divisors G now marked with 0, this is in fact a \mathcal{A} -stable family. □

Theorem 8.6. *The stable limit of a family of irreducible \mathcal{A} -stable elliptic surfaces is an \mathcal{A} -broken stable elliptic surface. In particular, the compact moduli stack $\mathcal{E}_{v,\mathcal{A}}$ parametrizes \mathcal{A} -broken stable elliptic surfaces.*

Proof. Every step of the proof of Theorem 8.4 produces a central fiber which is a broken elliptic surface. □

Corollary 8.7. *For any \mathcal{A} and \mathcal{B} within the same chamber, $\mathcal{E}_{v,\mathcal{A}} \cong \mathcal{E}_{v,\mathcal{B}}$.*

Proof. The walls of type W_I , W_{II} and W_{III} describe precisely when the log canonical divisor is nef rather than ample. Within a chamber $K_X + S + F_{\mathcal{A}}$ is ample if and only if $K_X + S + F_{\mathcal{B}}$ is ample and the log canonical models are the same. It follows that $\mathcal{E}_{v,\mathcal{A},n}^m(T) = \mathcal{E}_{v,\mathcal{B},n}^m(T)$ for any normal base T and so $\mathcal{E}_{v,\mathcal{A}} \cong \mathcal{E}_{v,\mathcal{B}}$ by Proposition A.7. \square

9. REDUCTION MORPHISMS

We begin by reviewing the notion of reduction morphisms present in the work of [Has03].

9.1. Hassett's moduli space. Recall the moduli spaces $\overline{\mathcal{M}}_{g,\mathcal{A}}$, parametrizing genus g curves with r marked points weighted by a weight vector $\mathcal{A} = (a_1, \dots, a_r)$ were defined in [Has03]. Hassett studied what happened as one lowers the weight vector \mathcal{A} . Namely, the following theorem guarantees the existence of reduction morphisms on the level of moduli spaces.

Theorem 9.1. [Has03, Theorem 4.1] *Fix g and n and let $\mathcal{A} = (a_1, \dots, a_r)$ and $\mathcal{B} = (b_1, \dots, b_r)$ two collections of admissible weights and suppose that $\mathcal{A} \leq \mathcal{B}$. Then there exists a natural birational reduction morphism*

$$\overline{\mathcal{M}}_{g,\mathcal{B}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}.$$

Given an element $(C, p_1, \dots, p_r) \in \overline{\mathcal{M}}_{g,\mathcal{B}}$, the image in $\overline{\mathcal{M}}_{g,\mathcal{A}}$ is obtained by collapsing components of C along which $K_C + a_1 p_1 + \dots + a_r p_r$ fails to be ample.

We will construct analogous reduction morphisms on the moduli spaces $\mathcal{E}_{v,\mathcal{A}}$ and their universal families which are compatible with the reduction morphisms of Hassett in the following way. The image curve $(C, f_*(F_{\mathcal{A}}))$ is naturally an \mathcal{A} -weighted curve in the sense of Hassett. We obtain a natural forgetful morphism from $\mathcal{E}_{v,\mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ for all $0 \leq a \leq 1$ (see Corollary 9.3) and the reduction morphisms (Theorem 9.4) will commute with Hassett's reduction morphisms above.

9.2. Preliminary results. Let $(f : X \rightarrow C, S + F_{\mathcal{A}})$ be an \mathcal{A} -broken elliptic surface. Denote by $D_{\mathcal{A}} := f_*(F_{\mathcal{A}})$ the weighted divisor on C corresponding to the weighted marked fibers of $f : X \rightarrow C$. Then $(C, D_{\mathcal{A}})$ is a weighted curve in the sense of Hassett.

Lemma 9.2. *Let $(X, S + F_{\mathcal{B}}) \xrightarrow{f} C \xrightarrow{q} B$ be a flat family of \mathcal{B} -stable elliptic surfaces over a base B . Denoting the composition $p = q \circ f$, then the formation of $p_*(f^* \omega_q(D_{\mathcal{A}})^{[m]})$ commutes with base change for any $\mathcal{A} \leq \mathcal{B}$ and $m \geq 1$.*

Proof. By Lemma 6.3, $p_*(f^* \omega_q(D_{\mathcal{B}})^{[m]}) = q_* f_* f^* \omega_q(D_{\mathcal{B}})^{[m]} = q_* \omega_q(D_{\mathcal{B}})^{[m]}$, and the latter commutes with base change by Proposition 3.3 of [Has03]. \square

First, we show the base curve of our weighted elliptic surface pairs are weighted stable curves in the sense of Hassett, so we can use these spaces to gain understanding of $\mathcal{E}_{v,\mathcal{A}}$.

Corollary 9.3. *There is a natural forgetful morphism $\mathcal{E}_{v,\mathcal{A}} \rightarrow \overline{\mathcal{M}}_{g,\mathcal{A}}$ given by sending a family of \mathcal{A} -broken stable elliptic surfaces $p : (f : X \rightarrow C, S + F_{\mathcal{A}}) \rightarrow B$ to the family of \mathcal{A} -weighted stable curves $q : (C, D_{\mathcal{A}}) \rightarrow B$.*

Proof. By Lemma 9.2, the formation of $p_*(f^*\omega_q(D_{\mathcal{A}})^{[m]}) = q_*\omega_q(D_{\mathcal{A}})^{[m]}$ commutes with base change. Therefore it suffices to check that $(C_b, (D_{\mathcal{A}})_b)$ is an \mathcal{A} -stable curve for each $b \in B$ and this is Corollary 5.12. \square

9.3. Reduction morphisms. We are now ready to state and prove our main theorem on reduction morphisms for moduli of elliptic surfaces analogous to [Has03, Theorem 4.1].

Theorem 9.4. *Let \mathcal{A} and \mathcal{B} be rational tuples such that $\mathcal{A} \leq \mathcal{B}$. Suppose that $\mathcal{A}(t)$ never lies on a Type W_{II} wall for $t > 0$ (see Remark 1). Then there exists a reduction morphisms $\rho_{\mathcal{A},\mathcal{B}} : \mathcal{E}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$. If we further suppose that $[\mathcal{A}] = [\mathcal{B}]$, then there exists a compatible $\tilde{\rho}_{\mathcal{A},\mathcal{B}} : \mathcal{U}_{v,\mathcal{B}} \rightarrow \mathcal{U}_{v,\mathcal{A}}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{U}_{v,\mathcal{B}} & \xrightarrow{\tilde{\rho}_{\mathcal{A},\mathcal{B}}} & \mathcal{U}_{v,\mathcal{A}} \\ \downarrow & & \downarrow \\ \mathcal{E}_{v,\mathcal{B}} & \xrightarrow{\rho_{\mathcal{A},\mathcal{B}}} & \mathcal{E}_{v,\mathcal{A}} \end{array}$$

All of the above reduction morphisms commute via the forgetful morphism of Corollary 9.3 with the reduction morphisms for Hassett space.

Remark 9.5. The condition $[\mathcal{A}] = [\mathcal{B}]$ just means that $a_i = 1$ if and only if $b_i = 1$. We consider the case when $a_i < b_i = 1$ in Proposition 10.7.

Proof. The proof that $\rho_{\mathcal{A},\mathcal{B}}$ is a morphism is modeled off of the proof of [Has03, Theorem 4.1]. Let $\mathcal{A} = (a_1, \dots, a_r)$ and $\mathcal{B} = (b_1, \dots, b_r)$ be so that $\mathcal{A} \leq \mathcal{B}$. Denote $\mathcal{A}(t) := (1-t)\mathcal{A} + t\mathcal{B}$, where $t \in \mathbb{Q}$ and $0 \leq t \leq 1$.

With notation from the proof of Theorem 4.4, we define a natural transformation of pseudofunctors

$$\mathcal{E}_{v,\mathcal{B},n}^m(B) \rightarrow \mathcal{E}_{v,\mathcal{A},n}^m(B)$$

for a normal base scheme B that is compatible with base change. This will lead to a morphism of moduli spaces $\rho_{\mathcal{A},\mathcal{B}} : \mathcal{E}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$ by Proposition A.6. There are necessarily finitely many $t_0 = 0, t_1, \dots, t_{k-1}, t_k = 1$ so that $\mathcal{A}(t_j)$ lie on walls and for any $t \neq t_j$, weights $\mathcal{A}(t)$ are not on any wall. It is clear that the weight vectors $\mathcal{A}(t_j) \leq \mathcal{A}(t_{j+1})$ satisfy the hypothesis of the theorem so it suffices to construct reduction morphisms $\rho_{\mathcal{A}(t_j),\mathcal{A}(t_{j+1})}$ so that

$$\rho_{\mathcal{A},\mathcal{B}} = \rho_{\mathcal{A}(t_0),\mathcal{A}(t_1)} \circ \dots \circ \rho_{\mathcal{A}(t_{k-1}),\mathcal{A}(t_k)}$$

Therefore we may assume that $\mathcal{A}(t)$ does not lie on a wall for any $t \neq 0, 1$, and that \mathcal{A} is either in a chamber or on a wall of type W_{I} or W_{III} . Writing $\mathcal{A}(t) = (a_1(t), \dots, a_r(t))$ this means explicitly that for all $0 < t < 1$,

- i) $a_j(t) \neq \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}$ (there are no type W_{I} walls);
- ii) there is no subset $\{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ such that $a_{i_1}(t) + \dots + a_{i_k}(t) = 1$ (there are no type W_{II} walls);
- iii) $K_X + S + F_{\mathcal{A}(t)}$ is big on every irreducible component of every \mathcal{B} -broken stable elliptic surface $(X \rightarrow C, S + F_{\mathcal{B}})$ (there are no type W_{III} walls).

Let $\pi : (f : X \rightarrow C, S + F_{\mathcal{B}}) \rightarrow B$ be a family of \mathcal{B} -broken elliptic surfaces over a normal base B . By our above assumption, $K_X + S + F_{\mathcal{A}(t)}$ is ample for $t > 0$ and $K_X + S + F_{\mathcal{A}(0)} = K_{X/B} + S + F_{\mathcal{A}}$ is π -nef and \mathbb{Q} -Cartier. By Proposition 6.2, $K_{X/B} + S + F_{\mathcal{A}}$ is π -semiample. Then we can write

$$\begin{aligned} C' &= \text{Proj}_B \left(\bigoplus_m \pi_* f^* \omega_{C/B}(mnD_{\mathcal{A}}) \right) \\ X' &= \text{Proj}_B \left(\bigoplus_m \pi_* \mathcal{O}_X(mn(K_{X/B} + S + F_{\mathcal{A}})) \right) \end{aligned}$$

where $D_{\mathcal{A}} = f_* F_{\mathcal{A}}$ and n is a large enough integer such that $nD_{\mathcal{A}}$ and $n(K_{X/B} + S + F_{\mathcal{A}})$ are Cartier.

There are stabilization maps $C \rightarrow C'$ and $X \dashrightarrow X'$. It follows from the basechange results Theorem 7.8 and Lemma 9.2 that X' and C' are families of \mathcal{A} -broken stable elliptic surfaces and \mathcal{A} -weighted pointed stable curves respectively. By Lemma 8.5, there is a map $f' : X' \rightarrow C'$ making $\pi' : (f' : X' \rightarrow C', S + F_{\mathcal{B}}) \rightarrow B$ a family of \mathcal{A} -broken stable elliptic surfaces over B . Since the construction of $\pi_* f^* \omega_{C/B}(mnD_{\mathcal{A}})$ and $\pi_* \mathcal{O}_X(mn(K_{X/B} + S + F_{\mathcal{A}}))$ commute with basechange by Theorem 7.8 and Lemma 9.2, it follows that the construction of the family of \mathcal{A} -broken stable elliptic surfaces is functorial in B . Furthermore, it is clear that the map on closed points, $\mathcal{E}_{v,\mathcal{B},n}^m(k) \rightarrow \mathcal{E}_{v,\mathcal{A},n}^m(k)$ is dominant on each component by observing that it is dominant on the locus of irreducible elliptic surfaces. This induces the required morphism

$$\rho_{\mathcal{A},\mathcal{B}} : \mathcal{E}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{A}}.$$

Next, we show existence of the morphism $\tilde{\rho}_{\mathcal{A},\mathcal{B}}$ on the level of universal families under the assumption $a_i = 1$ if and only if $b_i = 1$. In this case, there are no type W_I transformations from twisted to intermediate fibers so the rational map $X \dashrightarrow X'$ is actually a morphism $X \rightarrow X'$. The universal family $\mathcal{U}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{B}}$ is itself a family of \mathcal{B} -weighted stable elliptic surfaces. Therefore applying the above construction gives a family $\mathcal{Y} \rightarrow \mathcal{E}_{v,\mathcal{B}}$ of \mathcal{A} -stable elliptic surfaces with a morphism $\mathcal{U}_{v,\mathcal{B}} \rightarrow \mathcal{Y}$ over $\mathcal{E}_{v,\mathcal{B}}$. This induces the morphism $\rho_{\mathcal{A},\mathcal{B}}$ so that

$$\mathcal{Y} = \rho_{\mathcal{A},\mathcal{B}}^* \mathcal{U}_{v,\mathcal{A}}.$$

The composition $\mathcal{U}_{v,\mathcal{B}} \rightarrow \mathcal{Y} \rightarrow \mathcal{U}_{v,\mathcal{A}}$ gives the required $\tilde{\rho}_{\mathcal{A},\mathcal{B}}$.

The fact that these morphisms commute with the reduction morphisms for Hassett space is immediate since the forgetful map to the base curve is a morphism, and the family of base curves is stabilized by the linear series $\omega_{C/B}(nD_{\mathcal{A}})$ by Proposition 5.11 and Lemma 9.2. \square

Corollary 9.6. *The reduction morphisms $\rho_{\mathcal{A},\mathcal{B}}$ are surjective.*

Proof. This follows since $\rho_{\mathcal{A},\mathcal{B}}$ is a dominant morphism of proper stacks. \square

10. FLIPPING WALLS

Theorem 9.4, shows that there are reduction morphisms

$$\rho_{\mathcal{A},\mathcal{B}} : \mathcal{E}_{v,\mathcal{B}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$$

whenever $\mathcal{A}(t) := (1-t)\mathcal{A} + t\mathcal{B}$ never crosses a type W_{II} wall for $t \in (0, 1]$. The key point is that if $\mathcal{A}(t_0)$ is a type W_{II} wall for $t_0 \in (0, 1]$ and $t_{\pm} := t_0 \pm \epsilon$ for $0 < \epsilon \ll 1$, then

$$K_X + S + F_{\mathcal{A}(t_-)}$$

is *not necessarily* \mathbb{Q} -Cartier where $(f : X \rightarrow C, S + F_{\mathcal{A}(t_0)})$ is an $\mathcal{A}(t_0)$ -stable elliptic surface. Therefore it no longer makes sense to construct the $\mathcal{A}(t_-)$ -stable model as a Proj of the section ring.

Rather, to construct the $\mathcal{A}(t_-)$ -stable model from $(X, S + F_{\mathcal{A}(t_0)})$, we need to first perform a log resolution to make the log canonical divisor \mathbb{Q} -Cartier before running the steps of the minimal model program. Therefore, across type W_{II} walls, we obtain morphisms resembling *flips* (see Figure 13).

We fix some notation. Let t_0, t_{\pm} be as above where $\mathcal{A}_0 := \mathcal{A}(t_0)$ is on a wall of type W_{II} and $\mathcal{A}_- := \mathcal{A}(t_-) < \mathcal{A}_0 < \mathcal{A}(t_+) =: \mathcal{A}_+$ so that \mathcal{A}_{\pm} are in the interiors of chambers. We will use $(X_0, S_0 + F_{\mathcal{A}_0})$ and $(X_{\pm}, S_{\pm} + F_{\mathcal{A}_{\pm}})$ to denote \mathcal{A}_0 -stable and \mathcal{A}_{\pm} -stable elliptic surfaces respectively.

Theorem 10.1. *There exist morphisms $\tilde{\epsilon}_{\mathcal{A}_0}^-, \epsilon_{\mathcal{A}_0}^-$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{U}_{v,\mathcal{A}_-} & \xrightarrow{\tilde{\epsilon}_{\mathcal{A}_0}^-} & \mathcal{U}_{v,\mathcal{A}_0} \\ \downarrow & & \downarrow \\ \mathcal{E}_{v,\mathcal{A}_-} & \xrightarrow{\epsilon_{\mathcal{A}_0}^-} & \mathcal{E}_{v,\mathcal{A}_0} \end{array}$$

Proof. This proof is analogous to the proof of Theorem 9.4. Under these assumptions $K_{X_-} + S_- + F_{\mathcal{A}_0}$ is a semiample \mathbb{Q} -Cartier divisor and the \mathcal{A}_0 -stable model is

$$\text{Proj} \left(\bigoplus_{k \geq 0} H^0(X_-, \mathcal{O}_X(km_0(K_{X_-} + S_- + F_{\mathcal{A}_0}))) \right)$$

where m_0 is the index. Thus it suffices to prove a vanishing result analogous to Theorem 7.1.

Lemma 10.2. *In this situation, $H^i(\mathcal{O}_X(m(K_{X_-} + S_- + F_{\mathcal{A}_0}))) = 0$ for $i > 0$ and m large and divisible.*

Proof. We consider the irreducible components of X_- . There are three types of components:

- (a) a pseudoelliptic whose section was contracted at the wall \mathcal{A}_0 ;
- (b) a component along which a pseudoelliptic from case (a) is attached;
- (c) a component not in either of the above cases.

The pair $(X_-, S_- + F_{\mathcal{A}_0})$ is slc and the linear series $|K_{X_-} + S_- + F_{\mathcal{A}_0}|$ is semi-ample by Proposition 6.2. It induces a morphism $g : X_- \rightarrow X_0$ which is necessarily an isomorphism on the components in cases (a) and (c) above.

Suppose X' is a component in case (b). Then it is attached to a pseudoelliptic Z in case (a) along a fiber component G . As explained in La Nave (see Section 4.3 and Theorem 7.1.2 in [LN]), the contraction of the section of a component to form Z at the wall \mathcal{A}_0 may be a log flipping contraction inside of the total space of a one parameter degeneration with central fiber X_- . In this case, Z is a type I pseudoelliptic attached along an irreducible pseudofiber G to an intermediate (pseudo)fiber $G \cup A$ on X' (see Figure 13). The coefficient $\text{Coeff}(A, F_{\mathcal{A}})$ given by the sum of weights of fibers on Z as can be seen from La Nave's local toric model and the morphism $g : X_- \rightarrow X_0$ contracts A . In particular $\text{Coeff}(A, F_{\mathcal{A}_0}) = 1$.

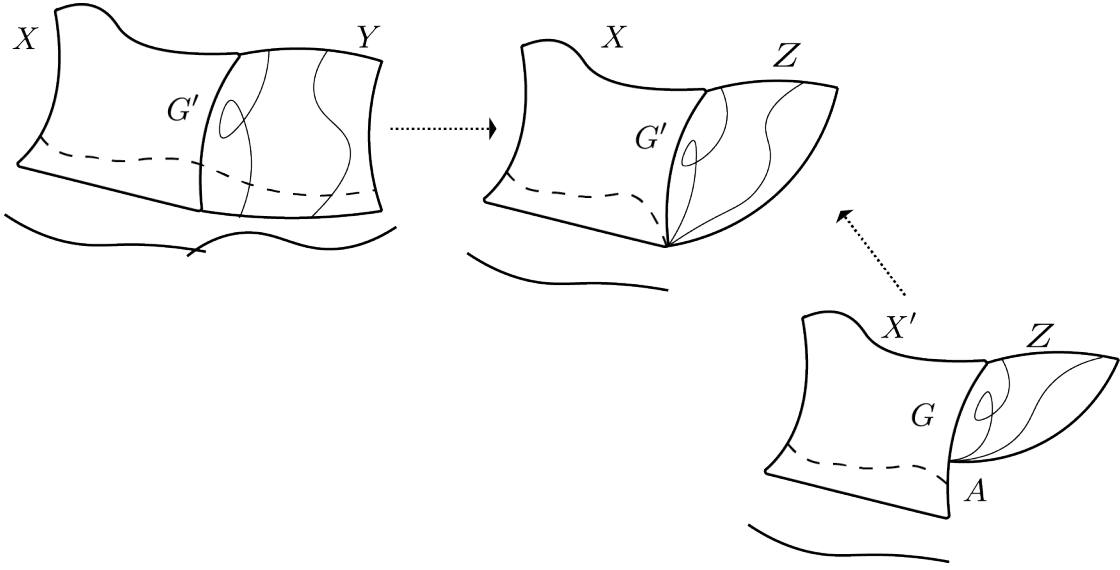


FIGURE 13. From left to right, the \mathcal{A}_+ , \mathcal{A}_0 and \mathcal{A}_- stable models. The sum of the weights of the marked pseudofibers on Z is equal to the coefficient of A in $F_{\mathcal{A}}$.

Thus $g : X_- \rightarrow X_0$ is precisely the contraction of these rational curves A produced by La Nave's flips. Denote $S_- + F_{\mathcal{A}_0} = \Delta$. Then by Proposition 6.8,

$$R^1 g_* \mathcal{O}_{X_-}(m(K_{X_-} + \Delta)) = 0.$$

On the other hand, $g_* \mathcal{O}_{X_-}(m(K_{X_-} + \Delta)) = \mathcal{O}_{X_0}(m(K_{X_0} + g_* \Delta))$ by Proposition 6.4, since $g_*^{-1} g_* \Delta + \text{Exc}(g) = \Delta$ as each curve A appears with coefficient 1. Therefore

$$H^1(X_-, \mathcal{O}_{X_-}(m(K_{X_-} + \Delta))) = H^1(X_0, \mathcal{O}_{X_0}(m(K_{X_0} + g_* \Delta))) = 0$$

since $(X_0, g_* \Delta) = (X_0, S_0 + F_{\mathcal{A}_0})$ is the \mathcal{A}_0 -stable model. \square

Now we can proceed as in the proof of Theorem 9.4: let $\pi : (X_- \rightarrow C, S_- + F_{\mathcal{A}_-}) \rightarrow B$ be an \mathcal{A}_- -weighted stable family of elliptic surfaces over a normal base B . Then the construction

of

$$\mathrm{Proj}_B \left(\bigoplus_k \pi_* \mathcal{O}_{X_-}(km_0(K_{X_-} + S_- + F_{\mathcal{A}_-})) \right)$$

commutes with base change and produces a family $\pi_0 : (X_0 \rightarrow C, S_0 + F_{\mathcal{A}_0})$ of \mathcal{A}_0 -stable elliptic surfaces and realizes the morphism $\epsilon_{\mathcal{A}_0}^-$. Applying this construction to $B = \mathcal{E}_{v, \mathcal{A}_-}$ with the universal family yields $\tilde{\epsilon}_{\mathcal{A}_0}^-$. \square

Remark 10.3. Note that in the above construction, the \mathcal{A}_0 -stable family $(X_0 \rightarrow C, S_0 + F_{\mathcal{A}_0})$ associated to the \mathcal{A}_- -stable family $(X_- \rightarrow C, S_- + F_{\mathcal{A}_-})$ has the same base curve C . This is because a marked curve is \mathcal{A}_0 -stable if and only if it is \mathcal{A}_- -stable where \mathcal{A}_0 is one of the walls for the space of weighted stable curves. That is, the reduction morphism $\mathcal{M}_{g, \mathcal{A}_0} \rightarrow \mathcal{M}_{g, \mathcal{A}_-}$ is a canonical isomorphism. In particular there is a commutative diagram

$$\begin{array}{ccccc} \mathcal{E}_{v, \mathcal{A}_-} & \xrightarrow{\epsilon_{\mathcal{A}_0}^-} & \mathcal{E}_{v, \mathcal{A}_0}^m & \xleftarrow{\epsilon_{\mathcal{A}_0}^+} & \mathcal{E}_{v, \mathcal{A}_+} \\ \downarrow & & \downarrow & & \downarrow \\ \mathcal{M}_{g, \mathcal{A}_-} & \xlongequal{\quad} & \mathcal{M}_{g, \mathcal{A}_0} & \xleftarrow{\quad} & \mathcal{M}_{g, \mathcal{A}_+} \end{array}$$

showing compatibility with the reduction morphisms on Hassett spaces.

Let $\tilde{\epsilon}_{\mathcal{A}_0}^+ := \tilde{\rho}_{\mathcal{A}_+, \mathcal{A}}$ and $\epsilon_{\mathcal{A}_0}^+ := \rho_{\mathcal{A}_+, \mathcal{A}}$ be the reduction morphisms of the previous section. Then we have a commuting diagram

$$\begin{array}{ccccc} \mathcal{U}_{v, \mathcal{A}_-} & & & & \mathcal{U}_{v, \mathcal{A}_+} \\ & \searrow \tilde{\epsilon}_{\mathcal{A}_0}^- & & \swarrow \tilde{\epsilon}_{\mathcal{A}_0}^+ & \\ & \mathcal{U}_{v, \mathcal{A}_0} & & & \\ & \downarrow & & & \\ \mathcal{E}_{v, \mathcal{A}_-} & & & & \mathcal{E}_{v, \mathcal{A}_+} \\ & \searrow \epsilon_{\mathcal{A}_0}^- & \downarrow & \swarrow \epsilon_{\mathcal{A}_0}^+ & \\ & \mathcal{E}_{v, \mathcal{A}_0} & & & \end{array}$$

We want to compare \mathcal{A}_+ -, \mathcal{A}_0 -, and \mathcal{A}_- -stable families of elliptic surfaces over the same base B . To do this, it is natural to consider the fiber product

$$\mathcal{E}_{v, \mathcal{A}_-} \times_{\mathcal{E}_{v, \mathcal{A}_0}} \mathcal{E}_{v, \mathcal{A}_+} =: \mathfrak{F}.$$

Pulling back the universal families gives a commutative diagram

$$\begin{array}{ccc}
 \mathfrak{U}_- & & \mathfrak{U}_+ \\
 & \searrow & \swarrow \\
 & \mathfrak{U}_0 & \\
 & \downarrow & \\
 & \mathfrak{F} &
 \end{array}$$

Then a map $B \rightarrow \mathfrak{F}$ is equivalent to a commutative diagram

$$\begin{array}{ccc}
 X_- & & X_+ \\
 & \searrow & \swarrow \\
 & X_0 & \\
 & \downarrow & \\
 & B &
 \end{array}$$

of compatible families $X_0, X_{\pm} \rightarrow B$ of \mathcal{A}_0 -stable (resp. \mathcal{A}_{\pm} -) stable elliptic surfaces.

We show that the diagram

$$\begin{array}{ccc}
 \mathfrak{U}_- & & \mathfrak{U}_+ \\
 & \searrow & \swarrow \\
 & \mathfrak{U}_0 &
 \end{array}$$

is a *universal flip* in the following sense:

Proposition 10.4. *For any normal and irreducible base B and map $B \rightarrow \mathfrak{F}$ with generic point mapping to the interior of the moduli space, the induced diagram*

$$\begin{array}{ccc}
 X_- & \xleftarrow{\varphi} & X_+ \\
 & \searrow g_- & \swarrow g_+ \\
 & X_0 &
 \end{array}$$

is a $(K_{X_+} + S_+ + F_{\mathcal{A}_-})$ -flip of the total spaces.

Proof. Let $V \subset B$ be the open locus over which the elliptic surfaces are irreducible and let $Z = B \setminus V$. By assumption V is nonempty and the morphisms $X_- \rightarrow X_0$ and $X_+ \rightarrow X_0$ are isomorphisms over V . Thus the exceptional locus $\text{Exc}(\varphi)$ lies over Z . On each fiber over Z , the map $X_+ \rightarrow X_0$ contracts the section of a pseudoelliptic component and $X_- \rightarrow X_0$ contracts a curve in an attaching fiber. Therefore the $\text{Exc}(\varphi)$ is codimension at least 2.

We need to show that $-(K_{X_+} + S_+ + F_{\mathcal{A}_-})$ is g_+ -ample and $K_{X_-} + \varphi_*(S_+ + F_{\mathcal{A}_-})$ is g_+ -ample. Note that $\varphi_*(S_+ + F_{\mathcal{A}_-}) = S_- + F_{\mathcal{A}_-}$, where by abuse of notation, we write $F_{\mathcal{A}}$ for \mathcal{A} -weighted fibers on any of the birational models. Since g_- and g_+ are proper, relative ampleness is a fiberwise condition. Thus it suffices to check this after pulling back to a

smooth curve $B' \rightarrow B$ so without loss of generality, we may take B to be an irreducible smooth curve so that $V = B \setminus \{p\}$.

In this case $X_+ \rightarrow X_0$ is the contraction of the section in a component of the central fiber $(X_+)_p$. It is then proven in [LN, Theorem 7.1.2] that $X_+ \rightarrow X_0$ is a flipping contraction induced by $K_{X_+} + S_+ + F_{\mathcal{A}_-}$ with log flip $X_- \rightarrow X_0$. \square

Corollary 10.5. *The morphism $\epsilon_{\mathcal{A}_0}^-$ is an isomorphism.*

Proof. $\epsilon_{\mathcal{A}_0}^-$ is a proper bijection and our moduli spaces are normal. \square

Remark 10.6. Since we normalize the moduli spaces, we make no claims about the infinitesimal structure of $\epsilon_{\mathcal{A}_0}^-$. Indeed the deformation theories of \mathcal{A}_0 and \mathcal{A}_- broken elliptic surfaces may be very different.

10.1. The “wall” at $a = 1$. In this section we discuss an analogous behavior to the flipping morphism $\epsilon_{\mathcal{A}_0}^- : \mathcal{E}_{v,\mathcal{A}_-}^m \rightarrow \mathcal{E}_{v,\mathcal{A}_0}^m$ that occurs in the limit as a coefficient $a \rightarrow 1$.

Indeed if we take $X_- = X' \cup Z$ as in the proof of Theorem 10.1 so that X' is an elliptic component, Z is a pseudoelliptic component of type I attached to X' along an intermediate fiber $G \cup A$, then we saw that the morphism $\epsilon_{\mathcal{A}_0}^-$ contracts the fiber component A . Locally on X' around the fiber $G \cup A$, this contraction of A is the transition from an intermediate to a twisted fiber Section 5. In both cases, this contraction occurs when the intermediate fiber components G and E are both marked with coefficient $a = 1$ and in both cases, this induces a morphism on moduli spaces:

Proposition 10.7. *Let $\mathcal{B} = (b_1, \dots, b_r)$ and fix j such that $b_j = 1$. Let $\mathcal{A} < \mathcal{B}$ be a weight vector with $a_i = b_i$ for $i \neq j$ and $a_j = 1 - \epsilon$ where $0 < \epsilon \ll 1$. Then there are morphisms $\theta_j : \mathcal{E}_{v,\mathcal{A}} \rightarrow \mathcal{E}_{v,\mathcal{B}}$ and $\tilde{\theta}_j : \mathcal{U}_{v,\mathcal{A}} \rightarrow \mathcal{U}_{v,\mathcal{B}}$ making the following diagram commute:*

$$\begin{array}{ccc} \mathcal{U}_{v,\mathcal{A}} & \xrightarrow{\tilde{\theta}_j} & \mathcal{U}_{v,\mathcal{B}} \\ \downarrow & & \downarrow \\ \mathcal{E}_{v,\mathcal{A}} & \xrightarrow{\theta_j} & \mathcal{E}_{v,\mathcal{B}} \end{array}$$

Proof. Since we are taking $\epsilon \ll 1$, then $K_X + S + F_{\mathcal{B}}$ is a nef \mathbb{Q} -Cartier divisor on a \mathcal{A} -stable elliptic surface $(X \rightarrow C, S + F_{\mathcal{A}})$. Therefore $K_X + S + F_{\mathcal{B}}$ is semiample and by the results of Section 5, there are two possibilities for the Iitaka map $\varphi := \varphi_{m(K_X + S + F_{\mathcal{B}})} : X \rightarrow X'$ depending on the fiber F_j whose coefficient is changing:

- the fiber F_j is a smooth or stable fiber (type I_n) so that the birational model does *not* change when $b_j = 1$ and φ is the identity;
- the fiber F_j is not type I_n so that it is an intermediate fiber given by a union $A \cup E$ of a reduced component A and a nonreduced component E . The Iitaka map φ is the contraction of E .

In the first case there is nothing to prove. In the second,

$$R^1\varphi_*(\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$$

by Proposition 6.8 and $\varphi_*(\mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = \mathcal{O}_{X'}(m(K_{X'} + \varphi_*(S + F_{\mathcal{A}})))$ by Proposition 5.4. It follows that $H^1(X, \mathcal{O}_X(m(K_X + S + F_{\mathcal{A}}))) = 0$ by the Leray spectral sequence.

Now if $\pi : (X \rightarrow C, S + F_{\mathcal{A}}) \rightarrow B$ is a family of \mathcal{A} -stable elliptic surfaces, then as in the construction of reduction morphisms and flipping morphisms,

$$\mathrm{Proj}_B \left(\bigoplus_k \pi_* \mathcal{O}_X(km_0(K_X + S + F_{\mathcal{B}})) \right)$$

gives a family \mathcal{B} -stable elliptic surfaces over B . This construction is compatible with base change by the above vanishing and induces the required morphism θ_j .

The morphism $\tilde{\theta}_j$ is induced by applying the above to the universal family $\mathcal{U}_{v,\mathcal{A}} \rightarrow \mathcal{E}_{v,\mathcal{A}}$. \square

Corollary 10.8. *In the situation above, the morphism θ_j is inverse to the reduction morphism $\rho_{\mathcal{B},\mathcal{A}}$. In particular, $\mathcal{E}_{v,\mathcal{A}} \cong \mathcal{E}_{v,\mathcal{B}}$.*

Remark 10.9. As in Remark 10.6, the validity of the above corollary hinges on the fact that we are defining our moduli spaces to be the normalizations of the appropriate pseudofunctors. In general the deformation theories of \mathcal{A} -stable and \mathcal{B} -stable models might differ depending on the choice of functor of stable pairs and we can only hope for θ_j to be some type of partial normalization.

APPENDIX A. NORMALIZATIONS OF ALGEBRAIC STACKS

In this appendix, we justify the fact that we only work with normal base schemes throughout the paper. Specifically, the goal is to prove that in certain situations, the normalization of an algebraic stack is uniquely determined by its values on normal base schemes (Proposition A.7) and that a morphism between normalizations of algebraic stacks can be constructed by specifying it on the category of normal schemes (Proposition A.6). This material is probably well known but we include it here for lack of a suitable reference.

If X is a locally Noetherian scheme, the normalization $\nu : X^\nu \rightarrow X$ is defined as the normalization of X in its total ring of fractions. We denote by $|X|$ (resp. $|\mathcal{X}|$) the underlying topological space of points of a scheme (resp. algebraic stack). We begin with some facts about normalizations of schemes.

Lemma A.1. [Sta16, Tag 035Q] *Let X be a locally Noetherian scheme;*

- (1) *the normalization $X^\nu \rightarrow X$ is integral, surjective and induces a bijection on irreducible components;*
- (2) *for any normal scheme Z and morphism $Z \rightarrow X$ such that each irreducible component of Z dominates an irreducible component of X , there exists a unique factorization $Z \rightarrow X^\nu \rightarrow X$.*

Lemma A.2. [Sta16, Tag 07TD] *Let $X \rightarrow Y$ be a smooth morphism of locally Noetherian schemes. Let $Y^\nu \rightarrow Y$ be the normalization of Y . Then $X \times_Y Y^\nu \rightarrow X$ is the normalization of X .*

This motivates the following definitions:

Definition A.3. Let \mathcal{X} be a locally Noetherian algebraic stack. We say that \mathcal{X} is **normal** if there is a smooth surjection $U \rightarrow \mathcal{X}$ where U is a normal scheme. A **normalization** of \mathcal{X} is a representable morphism

$$\nu : \mathcal{X}^\nu \rightarrow \mathcal{X}$$

from an algebraic stack \mathcal{X}^ν such that for any scheme U and any smooth morphism $U \rightarrow \mathcal{X}$, the pullback $\mathcal{X}^\nu \times_{\mathcal{X}} U \rightarrow U$ is the normalization of U .

Lemma A.4. *Let \mathcal{X} be a locally Noetherian algebraic stack. Then a normalization $\nu : \mathcal{X}^\nu \rightarrow \mathcal{X}$ exists and it is unique up to unique isomorphism.*

Proof. The proof closely follows [Sta16, Tag 07U4] which proves the claim for algebraic spaces. Indeed let $R \rightrightarrows U$ be a smooth groupoid presentation for \mathcal{X} . Then by Lemma A.2 one sees that the pullback of R to U^ν under both morphisms is isomorphic to R^ν . One can then check as in *loc. cit.* that $R^\nu \rightrightarrows U^\nu$ is a smooth groupoid and define $\mathcal{X}^\nu = [U^\nu / R^\nu]$ with morphism to \mathcal{X} induced by $U^\nu \rightarrow U$ and $R^\nu \rightarrow R$.

Normality is local on the base in the smooth topology [Sta16, Tag 034F] so that for any scheme T and smooth morphism $T \rightarrow \mathcal{X}$, we can check normality of $T \times_{\mathcal{X}} \mathcal{X}^\nu$ by pulling back to the smooth cover $U \rightarrow \mathcal{X}$. Here the result follows from Lemma A.2. Finally uniqueness is clear from the construction. \square

Lemma A.5. *Let \mathcal{X} be a locally Noetherian algebraic stack, then;*

- (1) \mathcal{X}^ν is normal;
- (2) $\mathcal{X}^\nu \rightarrow \mathcal{X}$ is an integral surjection that induces a bijection on irreducible components;
- (3) for any normal algebraic stack \mathcal{Z} and a morphism $\mathcal{Z} \rightarrow \mathcal{X}$ such that every irreducible component of \mathcal{Z} dominates an irreducible component of \mathcal{X} , there exists a unique factorization $\mathcal{Z} \rightarrow \mathcal{X}^\nu \rightarrow \mathcal{X}$.

Proof. The proof follows the analogous result [Sta16, Tag 0BB4] for algebraic spaces. (1) is clear and (2) follows from Lemma A.1 and descent.

For (3) let $U \rightarrow \mathcal{X}$ be a smooth surjection and $R = U \times_{\mathcal{X}} U \rightrightarrows U$. Pulling back to \mathcal{Z} gives a smooth morphism $\mathcal{Y} := U \times_{\mathcal{X}} \mathcal{Z} \rightarrow \mathcal{Z}$. Let $U' \rightarrow \mathcal{Y}$ be a smooth cover of \mathcal{Y} by a scheme and U' . The composition $U' \rightarrow \mathcal{Z}$ is a smooth cover with groupoid presentation $R' : U' \times_{\mathcal{Z}} U' \rightrightarrows U'$ and a commutative square

$$\begin{array}{ccc} R' & \longrightarrow & R \\ \Downarrow & & \Downarrow \\ U' & \longrightarrow & U \end{array}$$

The conditions on $\mathcal{Z} \rightarrow \mathcal{X}$ imply that we can apply Lemma A.1 to obtain unique factorizations $R' \rightarrow R^\nu$ and $U' \rightarrow U^\nu$. By uniqueness, these morphisms are compatible with the groupoid data so that we get a unique factorization $\mathcal{Z} \rightarrow \mathcal{X}^\nu$ by descent. \square

Now we are ready for the main results of this appendix.

Proposition A.6. *Let \mathcal{X} and \mathcal{Y} be locally Noetherian algebraic stacks. Suppose that for each normal scheme T , there exist functors*

$$f_T : \mathcal{X}(T) \rightarrow \mathcal{Y}(T)$$

compatible with base change and such that the induced morphism on points $|f| : |\mathcal{X}| \rightarrow |\mathcal{Y}|$ is dominant on irreducible components. Then f_T induces a unique representable morphism

$$f^\nu : \mathcal{X}^\nu \rightarrow \mathcal{Y}^\nu.$$

Proof. Let $U \rightarrow \mathcal{X}$ be a smooth surjection from a scheme U and let $U^\nu \rightarrow U$ be the normalization. Then $U^\nu \rightarrow \mathcal{X}$ is an integral surjection that induces a bijection on irreducible components by Lemma A.5 (2). Let $\xi \in \mathcal{X}(U^\nu)$ be the object inducing this morphism. Then we have an object $f_T(\xi) \in \mathcal{Y}(U^\nu)$ inducing a morphism $U^\nu \rightarrow \mathcal{Y}$. By assumption this is compatible with the pullbacks to $R^\nu = U^\nu \times_{\mathcal{X}^\nu} \times U^\nu$ and thus induces a morphism $g : \mathcal{X}^\nu \rightarrow \mathcal{Y}$.

The map $|g| : |\mathcal{X}| \rightarrow |\mathcal{Y}|$ factors as

$$\begin{array}{ccc} |\mathcal{X}^\nu| & \xrightarrow{|f|} & |\mathcal{X}| \\ & \searrow |g| & \downarrow |f| \\ & & |\mathcal{Y}| \end{array}.$$

By Lemma A.5 (2) and the assumptions on $|f|$, $|g|$ is dominant on irreducible components. Therefore there is a unique factorization $f^\nu : \mathcal{X}^\nu \rightarrow \mathcal{Y}^\nu$ by Lemma A.5 (3). \square

Proposition A.7. *Let \mathcal{X} and \mathcal{Y} be separated locally Noetherian algebraic stacks. Suppose that for each normal scheme T , there is an isomorphism $f_T : \mathcal{X}(T) \cong \mathcal{Y}(T)$ compatible with base change. Then there is an isomorphism $f : \mathcal{X}^\nu \rightarrow \mathcal{Y}^\nu$.*

Proof. First let \mathcal{T} be a normal algebraic stack. Then there is a smooth cover $U \rightarrow \mathcal{T}$ where U is normal giving a groupoid presentation $R \rightrightarrows U$ of \mathcal{T} . Since normality is local in the smooth topology [Sta16, Tag 034F], R is normal and we have equivalences $\mathcal{X}(R) \cong \mathcal{Y}(R)$ and $\mathcal{X}(U) \cong \mathcal{Y}(U)$ compatible with base change by the two morphisms $R \rightrightarrows U$. By descent, this induces an equivalence $f_{\mathcal{T}} : \mathcal{X}(\mathcal{T}) \cong \mathcal{Y}(\mathcal{T})$ compatible with base change by a normal algebraic stack. Denote the inverse by $g_{\mathcal{T}}$.

By Proposition A.6 there exist morphisms $f : \mathcal{X}^\nu \rightarrow \mathcal{Y}^\nu$ and $g : \mathcal{Y}^\nu \rightarrow \mathcal{X}^\nu$ induced by f_T and its inverse. The map $\mathcal{X}^\nu \rightarrow \mathcal{X}$ is induced by an object $\xi \in \mathcal{X}(\mathcal{X}^\nu)$ and under the equivalence described in the preceding paragraph, $f_{\mathcal{X}^\nu}(\xi) \in \mathcal{Y}(\mathcal{X}^\nu)$ corresponds to the composition $\mathcal{X}^\nu \rightarrow \mathcal{Y}^\nu \rightarrow \mathcal{Y}$. Similarly, if $\xi' \in \mathcal{Y}(\mathcal{Y}^\nu)$ is the object inducing the normalization $\mathcal{Y}^\nu \rightarrow \mathcal{Y}$, then $g_{\mathcal{Y}^\nu}(\xi') \in \mathcal{X}(\mathcal{Y}^\nu)$ corresponds to the composition $\mathcal{Y}^\nu \rightarrow \mathcal{X}^\nu \rightarrow \mathcal{X}$.

By compatibility of the equivalences with pullbacks, we get that $g^*\xi = g_{\mathcal{Y}^\nu}(\xi')$ so that $\xi' = f_{\mathcal{Y}^\nu}g^*\xi = g^*f_{\mathcal{X}^\nu}\xi \in \mathcal{Y}(\mathcal{Y}^\nu)$. But the latter is the object corresponding to the composition

$$\mathcal{Y}^\nu \rightarrow \mathcal{X}^\nu \rightarrow \mathcal{Y}^\nu \rightarrow \mathcal{Y}.$$

Therefore $\nu \circ f \circ g = \nu$, i.e. the morphism $fg : \mathcal{Y}^\nu \rightarrow \mathcal{Y}^\nu$ commutes with the normalization $\mathcal{Y}^\nu \rightarrow \mathcal{Y}$.

Since the normalization factors uniquely through \mathcal{Y}^{red} , we may suppose that \mathcal{Y} is reduced. Then ν is an isomorphism over a dense open subset of each irreducible component of \mathcal{Y} . Therefore fg must agree with the identity over this dense open subset so $fg = \text{id}_{\mathcal{Y}^\nu}$, since \mathcal{Y}^ν is separated. Applying the same argument to \mathcal{X}^ν yields that $gf = \text{id}_{\mathcal{X}^\nu}$. \square

Remark A.8. Note that $\mathcal{X}^\nu(T)$ is *not* necessarily equal to $\mathcal{X}(T)$ for T normal even though \mathcal{X}^ν is uniquely determined by the values of $\mathcal{X}(T)$ for T normal. Indeed this fails even for schemes. For example the inclusion of the node of nodal curve has multiple lifts to the normalization. It is an interesting question to determine a functorial way to define the normalization of \mathcal{X} directly as a category fibered in groupoids over schemes without knowing a priori that \mathcal{X} is algebraic.

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E-mail address: kenneth_ascher@brown.edu

E-mail address: dbejleri@math.brown.edu